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GDDP: Generalized Dual Dynamic Programming Theory

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Abstract. This document presents theoretical considerations about the solution of dynamic optimization problems integrating the Benders Theory, the Dynamic Programming approach and the concepts of Control Theory. The so called Generalized Dual Dynamic Programming Theory (GDDP) can be considered as an extension of two previous approaches known as Dual Dynamic Programming (DDP): The first is the work developed by Pereira and Pinto [3–5], which was revised by Velásquez and others [8,9]. The second is the work developed by Read and others [2,6,7].

Keywords: Benders decomposition, dynamic programming, dual dynamic programming, control theory

1. Introduction

This paper analyzes the theory developed by Benders (BT) [1] applied to the solution of dynamic problems using the Dynamic Programming and the Control Theory approaches. We call this methodology Generalized Dual Dynamic Programming (GDDP) which is based on the chained application of BT to a multi-period optimization problem. Two previous works must be kept in mind as an initial reference, both were called Dual Dynamic Programming (DDP).

First, in a series of papers published starting in 1985 Pereira and Pinto [3–5] introduced a technique that made it possible to apply the methodology of Dynamic Programming (DP) to problems with multiple state variables without running into the "curse of dimensionality problem". Pereira and Pinto extended Benders' partition technique to the multiple-stage problem, which allowed the replacement of the discretization of the future cost function in the classic DP approach by a series of hyperplanes generated using BT. Velásquez and others [8,9] revised and adjusted the equations of the methodology proposed by Pereira. We call this final methodology DDP-P. The main difference between DDP-P and GDDP is that in the conceptual formulation DDP-P considers all the variables of the problem as state variables, while the GDDP makes a distinction between state variables and control variables. This distinction permits a more detailed algorithm in which the sub-problems are smaller than in the DDP-P.

Read and others developed the second work [2,6,7], which we call DDP-R. The conceptual formulation of the DDP-R makes the distinction between production (control) variables and stocks (state) variables. The DDP-R may be seen as dual to the classic DP approach in the sense that DP chooses an arbitrary grid of primal state vari-

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ables (stocks) and for each one finds the optimal decision and the implied shadow price (the marginal costs of stocks). DDP-R chooses a special grid of dual variables, values of the marginal costs of stocks in which the production decision changes and for each one finds the corresponding stock level in the primal space. The DDP-R solves parametrically a sub-problem in each period to produce points of the "supply function" for that period and, using backward recursion, combines these supply functions with the end-of-period function value to define an optimal strategy. The DDP-R has the "curse of dimensionality problem". The conceptual approach of the GDDP is similar to the DDP-R approach, in the sense that it maintains the difference between control variables and state variables; the fundamental difference is the form of evaluating the supply functions. In the GDDP the supply functions are built based on a set of hyperplanes generated using BT, the same way DDP-P does for the future cost functions.

The GDDP considers the solution of a general dynamic problem of the form:

$$
\begin{aligned}
\text{GDP:}\quad & \min z = \sum_{t \in \Theta(T)} \boldsymbol{c}_t^{\mathrm{T}} \boldsymbol{x}_t + \boldsymbol{d}_t^{\mathrm{T}} \boldsymbol{u}_t \\
& \text{subject to} \quad \boldsymbol{A}_t \boldsymbol{x}_t = \boldsymbol{b}_t - \boldsymbol{E}_{t-1} \boldsymbol{x}_{t-1} - \boldsymbol{B}_t \boldsymbol{u}_t \qquad \forall t \in \Theta(T), \\
& \qquad\qquad\quad \boldsymbol{G}_t \boldsymbol{u}_t = \boldsymbol{g}_t \qquad \forall t \in \Theta(T), \\
& \qquad\qquad\quad \boldsymbol{u}_t \in \boldsymbol{R}^+ \qquad \forall t \in \Theta(T), \\
& \qquad\qquad\quad \boldsymbol{x}_t \in \boldsymbol{R}^+ \qquad \forall t \in \Theta(T),
\end{aligned}
\tag{1}
$$

where the vector $\boldsymbol{x}_t$ represents the state variables and $\boldsymbol{u}_t$ the control variables. $\boldsymbol{A}_t$, $\boldsymbol{E}_t$, $\boldsymbol{B}_t$ and $\boldsymbol{G}_t$ are coefficients matrices, $\boldsymbol{b}_t$ and $\boldsymbol{g}_t$ are resources vectors, $\boldsymbol{c}_t$ and $\boldsymbol{d}_t$ are cost vectors, and $\Theta(T)$ the set of periods of the planning horizon composed of T time intervals.

The previous formulation may be appropriate for industrial linear systems in which the state variables vector is associated with the amount of stock held and the level of resources in the facilities, and the control vector is associated with the production and distribution of products through the supply-chain. The GDP family of optimization problems is used in the modeling of large supply-chains to support decision making at the tactical level in which the nonlinear characteristics can be linearized. The application of GDDP theory solves GDP as a coordinated sum of very simple problems. Initially, we present a summary of BT and DDP-P.

2. Previous mathematical formulations

2.1. *Benders' partition theory*

BT considers the problem P composed by two types of variables: $\boldsymbol{y}$, the coordination variables, and $\boldsymbol{x}$, the coordinates:

$$\begin{array}{ll} \text{P:} & \min z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + f(\boldsymbol{y}) \\ & \text{subject to} \quad \boldsymbol{F}_0(\boldsymbol{y}) = \boldsymbol{b}_0, \\ & \qquad\qquad\quad \boldsymbol{A}\boldsymbol{x} + \boldsymbol{F}(\boldsymbol{y}) = \boldsymbol{b}, \\ & \qquad\qquad\quad \boldsymbol{x} \in \boldsymbol{R}^+, \quad \boldsymbol{y} \in \boldsymbol{S}. \end{array} \tag{2}$$

BT restricts the model on $\boldsymbol{x}$ to be a linear problem, while imposing no conditions on $\boldsymbol{y}$. The $\boldsymbol{S}$ space to which $\boldsymbol{y}$ belongs may be continuous or discrete. Additionally, the functions $f(\boldsymbol{y})$ and $\boldsymbol{F}(\boldsymbol{y})$ may be nonlinear. The P problem may be decomposed in two coordinated problems: one, CY, on $\boldsymbol{y}$ and another, SP($\boldsymbol{y}$), on $\boldsymbol{x}$.

Benders proposes a solution of P by a hierarchical algorithm that works on two levels: the coordination level solves the problem CY and generates a sequence of $\boldsymbol{y}^k$ values; on the second level, $\boldsymbol{y}^k$ is used as a parameter of the sub-problem SP($\boldsymbol{y}$) to generate a sequence of feasible extreme points, π^k, and extreme rays, ω^k, of the dual feasible zone of SP($\boldsymbol{y}$) that are used to build in CY cutting planes.

Let us define the sub-problem SP($\boldsymbol{y}$) on $\boldsymbol{x}$ for a given value of $\boldsymbol{y}$

$$\begin{array}{ll} \text{SP}(\boldsymbol{y})\text{:} & \min Q(\boldsymbol{y}) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ & \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{F}(\boldsymbol{y}), \quad \boldsymbol{x} \in \boldsymbol{R}^+. \end{array} \tag{3}$$

The coordinator CY on $\boldsymbol{y}$ can be formulated as:

$$\begin{array}{llr} \text{CY:} & \min z = f(\boldsymbol{y}) + Q(\boldsymbol{y}) & \\ & \text{subject to} \quad \boldsymbol{F}_0(\boldsymbol{y}) = \boldsymbol{b}_0, & \boldsymbol{y} \in \boldsymbol{S}, \\ & \qquad Q(\boldsymbol{y}) \geqslant \left(\pi^k\right)^{\mathrm{T}}\left[\boldsymbol{b} - \boldsymbol{F}(\boldsymbol{y})\right] & \forall \boldsymbol{k} \in IT, \\ & \qquad 0 \geqslant \left(\omega^k\right)^{\mathrm{T}}\left[\boldsymbol{b} - \boldsymbol{F}(\boldsymbol{y})\right] & \forall \boldsymbol{k} \in IN, \end{array} \tag{4}$$

where π represents the vector of dual variables of the restrictions $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{F}(\boldsymbol{y})$, IT the set of iterations, ω an extreme ray on the π feasibility region and IN the set of iterations on which no feasibility was obtained.

CY includes two types of cuts. The first type, that we call optimality cutting planes (OCP), restricts the feasible zone of $\boldsymbol{y}$ in order to obtain the optimal $\boldsymbol{y}$; it has the following structure:

$$Q(\boldsymbol{y}) \geqslant \left(\pi^k\right)^{\mathrm{T}}\left[\boldsymbol{b} - \boldsymbol{F}(\boldsymbol{y})\right] \quad \forall \boldsymbol{k} \in IT. \tag{5}$$

The second type, that we call feasibility cutting planes (FCP), restricts the feasible zone of $\boldsymbol{y}$ in order to maintain feasible $\boldsymbol{x}$ in SP($\boldsymbol{y}$) and has the following structure

$$0 \geqslant \left(\omega^k\right)^{\mathrm{T}}\left[\boldsymbol{b} - \boldsymbol{F}(\boldsymbol{y})\right] \quad \forall \boldsymbol{k} \in IN. \tag{6}$$

For simplicity, in the following sections the mathematical formulation ignores the cuts FCP. This implies that it is only valid for problems where it is assured that a feasible $\boldsymbol{x}$ exists for any feasible $\boldsymbol{y}$. This is not a limitation of the theory. If it is necessary to include FCP, they may be included in a similar way as is done with the OCP cuts.

2.2. *Dual Dynamic Programming (DDP-P)*

Now, we describe the revised concepts of DDP-P that consider the following problem:

$$\begin{aligned} \text{DP:}\quad & \min z = \sum_{t\in\Theta(T)} \boldsymbol{c}_t^{\mathrm{T}}\boldsymbol{x}_t \\ & \text{subject to}\quad \boldsymbol{A}_t\boldsymbol{x}_t + \boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1} = \boldsymbol{b}_t \quad \forall t\in\Theta(T),\ \boldsymbol{x}_t\in\boldsymbol{R}^+. \end{aligned} \tag{7}$$

This structure corresponds to a problem of dynamic programming that includes only state variables ($\boldsymbol{x}_t$), with $\boldsymbol{x}_0$ as initial condition.

As a starting point the variables must be divided in two groups: a group of coordination variables (type $\boldsymbol{y}$) associated with stages between 1 and $T-1$, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{T-1}\}$, and the coordinated variables (type $\boldsymbol{x}$) associated with stage T, $\{\boldsymbol{x}_T\}$.

Based on the direct application of BT the coordinator model for the period $\{1, T-1\}$ is

$$\begin{aligned} \text{CD}_{T-1}:\quad & \min z = \sum_{t\in\Theta(T-1)} \boldsymbol{c}_t^{\mathrm{T}}\boldsymbol{x}_t + \alpha_T(\boldsymbol{x}_{T-1}) \\ & \text{subject to}\quad \boldsymbol{A}_t\boldsymbol{x}_t = \boldsymbol{b}_t - \boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1} && \forall t\in\Theta(T-1), \\ & \qquad\qquad\quad \boldsymbol{x}_t\in\boldsymbol{R}^+ && \forall t\in\Theta(T-1), \\ & \qquad\qquad\quad \alpha_T(\boldsymbol{x}_{T-1}) + \pi_T^k\boldsymbol{E}_{T-1}\boldsymbol{x}_{T-1} \geqslant \pi_T^{k\mathrm{T}}\boldsymbol{b}_T && \forall k\in IT(T-1), \end{aligned} \tag{8}$$

where $\alpha_t(\boldsymbol{x}_{t-1})$ represents the future cost function considering that the system is on state $\boldsymbol{x}_{t-1}$ at the beginning of period t, $IT(t)$ the number of cuts included in the coordinator CD_t, and π_t^k corresponds to the vector of dual variables of the restrictions $\boldsymbol{A}_t\boldsymbol{x}_t = \boldsymbol{b}_t - \boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1}^k$ obtained as a solution of the sub-problem $\text{SD}_t(\boldsymbol{x}_{t-1}^k)$. For the stage T the sub-problem $\text{SD}_T(\boldsymbol{x}_{T-1}^k)$ is defined as

$$\begin{aligned} \text{SD}_T\big(\boldsymbol{x}_{T-1}^k\big):\quad & \min \alpha_T\big(\boldsymbol{x}_{T-1}^k\big) = \boldsymbol{c}_T^{\mathrm{T}}\boldsymbol{x}_T \\ & \text{subject to}\quad \boldsymbol{A}_T\boldsymbol{x}_T = \boldsymbol{b}_T - \boldsymbol{E}_{T-1}\boldsymbol{x}_{T-1}^k,\ \boldsymbol{x}_T\in\boldsymbol{R}^+. \end{aligned} \tag{9}$$

Using a recursive approach, by induction [8,9] it can be demonstrated that the coordinator for each intermediate stage t (less than T) is

$$\begin{aligned} \text{CD}_t:\quad & \min z = \sum_{\tau\in\Theta(t)} \boldsymbol{c}_\tau^{\mathrm{T}}\boldsymbol{x}_\tau + \alpha_{t+1}(\boldsymbol{x}_t) \\ & \text{subject to}\quad \boldsymbol{A}_\tau\boldsymbol{x}_\tau = \boldsymbol{b}_\tau - \boldsymbol{E}_{\tau-1}\boldsymbol{x}_{\tau-1} && \forall \tau\in\Theta(t), \\ & \qquad\qquad\quad \boldsymbol{x}_\tau\in\boldsymbol{R}^+ && \forall \tau\in\Theta(t), \\ & \qquad\qquad\quad \alpha_{t+1}(\boldsymbol{x}_t) + \pi_{t+1}^{j}{}^{\mathrm{T}}\boldsymbol{E}_t\boldsymbol{x}_t \geqslant \sigma_{t+1}^j && \forall j\in IT(t), \end{aligned} \tag{10}$$

and the sub-problem for each intermediate stage t is

$$\begin{aligned} \text{SD}_t\big(\boldsymbol{x}_{t-1}^j\big):\quad & \min \alpha_t(\boldsymbol{x}_{t-1}^j) = \boldsymbol{c}_t^{\mathrm{T}}\boldsymbol{x}_t + \alpha_{t+1}(\boldsymbol{x}_t) \\ & \text{subject to}\quad \boldsymbol{A}_t\boldsymbol{x}_t = \boldsymbol{b}_t - \boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1}^j, && \boldsymbol{x}_t\in\boldsymbol{R}^+, \\ & \qquad\qquad\quad \alpha_{t+1}(\boldsymbol{x}_t) + \pi_{t+1}^{k}{}^{\mathrm{T}}\boldsymbol{E}_t\boldsymbol{x}_t \geqslant \sigma_{t+1}^k && \forall k\in IJ(t,j), \end{aligned} \tag{11}$$

where $IJ(t, j)$ represents the number of cuts included in the coordinator CD_t: when we solve the sub-problem $\mathrm{SD}_t(\boldsymbol{x}_{t-1}^j)$. σ_t^j is a constant value calculated as

$$\sigma_t^j = \begin{cases} \pi_T^{j\mathrm{T}} \boldsymbol{b}_T, & t = T, \\ \pi_t^{j\mathrm{T}} \boldsymbol{b}_t + \sum_{k=1,IJ(t,j)} \lambda_t^{k,j} \sigma_{t+1}^k, & t = 1, T-1, \end{cases} \tag{12}$$

where $\lambda_t^{k,j}$ represents the dual variable of the kth Benders cut in the sub-problem $\mathrm{SD}_t(\boldsymbol{x}_{t-1}^j)$. The index j represents the iteration of the coordinator of $\mathrm{SD}_t(\boldsymbol{x}_{t-1}^j)$ and the index k the cuts that have been included in the process of solution of $\mathrm{SD}_t(\boldsymbol{x}_{t-1}^j)$. The original problem DP corresponds to CD_1, the coordinator for the first stage.

The DDP-P approach implies the multilevel decomposition (in the time domain) of multiple problems that are decomposed using BT, generating new problems that again are decomposed using BT. This implies the nested use of BT.

3. Generalized Dual Dynamic Programming

The problem studied in DDP-P only considers state variables ($\boldsymbol{x}_t$). The problem GDP, presented in (1), considers a more detailed formulation in which the variables that couple two consecutive periods, related to the stock level, correspond to the state variables ($\boldsymbol{x}_t$), while the production and distribution variables in the period t, not involved directly in the dynamic relationship, may be thought as the control variables ($\boldsymbol{u}_t$).

The solution of GDP using BT considers a two-stage process. First, we define the state variable $\boldsymbol{x}_t$ as the coordination variables to go on to decouple the problem at a temporary level that refers to the control variable $\boldsymbol{u}_t$. In the second stage the DDP-P principles are used to solve the coordinator problem for $\boldsymbol{x}_t$

$$\begin{aligned} \text{CX:} \quad & \min z = \sum_{t \in \Theta(T)} \boldsymbol{c}_t^{\mathrm{T}} \boldsymbol{x}_t + \Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t) \\ & \text{subject to} \quad \min \Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t) = \boldsymbol{d}_t^{\mathrm{T}} \boldsymbol{u}_t \\ & \qquad\qquad\quad \text{subject to} \quad \boldsymbol{B}_t u_t = \boldsymbol{b}_t - \boldsymbol{E}_{t-1} \boldsymbol{x}_{t-1} - \boldsymbol{A}_t \boldsymbol{x}_t, \\ & \qquad\qquad\qquad\qquad\qquad \boldsymbol{G}_t u_t = \boldsymbol{g}_t, \\ & \qquad\qquad\qquad\qquad\qquad \boldsymbol{u}_t \in \boldsymbol{R}^+ \\ & \qquad\qquad\quad \forall t \in \Theta(T), \\ & \qquad\qquad\quad \boldsymbol{x}_t \in \boldsymbol{R}^+ \quad \forall t \in \Theta(T), \end{aligned} \tag{13}$$

where $\Omega_t(\boldsymbol{x}_{t-1,} \boldsymbol{x}_t)$ represents the optimum operation costs in the period t as a consequence of the border condition starting in the state $\boldsymbol{x}_{t-1}$ and finishing in the $\boldsymbol{x}_t$ and it corresponds to the objective function of the static operation sub-problems $\mathrm{SU}_t(\boldsymbol{x}_{t-1,} \boldsymbol{x}_t)$ defined as

$$\begin{aligned}\mathrm{SU}_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t):\quad &\min \Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)=\boldsymbol{d}_t^{\mathrm{T}}\boldsymbol{u}_t\\ &\text{subject to}\quad \boldsymbol{B}_t\boldsymbol{u}_t=\boldsymbol{b}_t-\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1}-\boldsymbol{A}_t\boldsymbol{x}_t,\ \boldsymbol{G}_t\boldsymbol{u}_t=\boldsymbol{g}_t,\ \boldsymbol{u}_t\in\boldsymbol{R}^+.\end{aligned}\tag{14}$$

The dual problem of $\mathrm{SU}_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)$ is

$$\begin{aligned}\mathrm{DSU}_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t):\quad &\max \Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)=\pi_t^{\mathrm{T}}[\boldsymbol{b}_t-\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1}-\boldsymbol{A}_t\boldsymbol{x}_t]+\delta_t^{\mathrm{T}}\boldsymbol{g}_t\\ &\text{subject to}\quad \pi_t^{\mathrm{T}}\boldsymbol{B}_t+\delta_t^{\mathrm{T}}\boldsymbol{G}_t\leqslant\boldsymbol{d}_t^{\mathrm{T}},\end{aligned}\tag{15}$$

where π_t represents the dual variables vector of the restrictions $\boldsymbol{B}_t\boldsymbol{u}_t=\boldsymbol{b}_t-\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1}-\boldsymbol{A}_t\boldsymbol{x}_t$ and δ_t is the dual variables vector of $\boldsymbol{G}_t\boldsymbol{u}_t=\boldsymbol{g}_t$.

Considering the decoupled cuts generated by each sub-problem $\mathrm{SU}_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)$ the coordinator CX is

$$\begin{aligned}\mathrm{CX}:\quad &\min z=\sum_{t\in\Theta(T)}\boldsymbol{c}_t^{\mathrm{T}}\boldsymbol{x}_t+\Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)\\ &\text{subject to}\quad \Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)+\left(\pi_t^k\right)^{\mathrm{T}}\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1}+\left(\pi_t^k\right)^{\mathrm{T}}\boldsymbol{A}_t\boldsymbol{x}_t\geqslant\theta_t\left(\pi_t^k,\delta_t^k\right)\\ &\qquad\forall t\in\Theta(T)\ \forall k\in IU,\end{aligned}\tag{16}$$

where IU represents the number of cuts generated for each sub-problem $SU_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)$ and $\theta_t(\pi,\delta)$ is a two argument t-index function that define a constant value

$$\theta_t(\pi,\delta)=\pi^{\mathrm{T}}\boldsymbol{b}_t+\delta^{\mathrm{T}}\boldsymbol{g}_t.\tag{17}$$

The coordinator CX is only integrated by Benders cuts and has a dynamic structure similar to the problem DP. We may solve it by using the DDP-P theory. The cuts that integrated the coordinator CX

$$\Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)+\left(\pi_t^k\right)^{\mathrm{T}}\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1}+\left(\pi_t^k\right)^{\mathrm{T}}\boldsymbol{A}_t\boldsymbol{x}_t\geqslant\theta_t\left(\pi_t^k,\delta_t^k\right)\quad\forall t\in\Theta(T)\ \forall k\in IU\tag{18}$$

will be called type 1 Benders cuts.

Following the backward approach we can apply BT for the last stage of the coordinator CX. The coordinator model CG_{T-1} for the period $\{1, T-1\}$ is

$$\begin{aligned}\mathrm{CG}_{T-1}:\quad &\min z=\sum_{t\in\Theta(T-1)}\left[\boldsymbol{c}_t^{\mathrm{T}}\boldsymbol{x}_t+\Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)\right]+\alpha_T(\boldsymbol{x}_{T-1})\\ &\text{subject to}\quad \Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)+\left(\pi_t^k\right)^{\mathrm{T}}\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1}+\left(\pi_t^k\right)^{\mathrm{T}}\boldsymbol{A}_t\boldsymbol{x}_t\\ &\qquad\geqslant\theta_t\left(\pi_t^k,\delta_t^k\right)\quad\forall t\in\Theta(T-1)\ \forall k\in IU,\\ &\qquad\boldsymbol{x}_t\in\boldsymbol{R}^+\quad\forall t\in\Theta(T-1),\\ &\qquad\alpha_T(\boldsymbol{x}_{T-1})+\sum_{k\in II(T,j)}\psi_T^{k,j}\left(\pi_T^k\right)^{\mathrm{T}}\boldsymbol{E}_{T-1}\boldsymbol{x}_{T-1}\\ &\qquad\geqslant\sum_{k\in II(T,j)}\psi_T^{k,j}\theta_T\left(\pi_T^k,\delta_T^k\right)\quad\forall j\in IJ(T-1),\end{aligned}\tag{19}$$

where $\alpha_{t+1}(\boldsymbol{x}_t)$ is the same future cost function defined in the DDP-P theory, $\psi_t^{k,j}$ the dual variable of the kth type 1 Benders cut for the period t obtained in the solution of the sub-problem $\mathrm{SG}_t(\boldsymbol{x}_{t-1}^j)$, $IJ(t)$ the number of cuts generated for the sub-problem $\mathrm{SG}_t(\boldsymbol{x}_{t-1}^j)$ and $II(t,j)$ the number of Benders cuts type 1 included in the sub-problem $\mathrm{SG}_t(\boldsymbol{x}_{t-1}^j)$.

The sub-problem $\mathrm{SG}_t(\boldsymbol{x}_{t-1}^j)$ for stage T is

$$\begin{aligned}\mathrm{SG}_T\big(\boldsymbol{x}_{T-1}^j\big):\quad & \min \alpha_T\big(\boldsymbol{x}_{T-1}^j\big) = \boldsymbol{c}_T^{\mathrm{T}}\boldsymbol{x}_T + \Omega_T(\boldsymbol{x}_{T-1},\boldsymbol{x}_T)\\ & \text{subject to}\quad \Omega_T(\boldsymbol{x}_{T-1},\boldsymbol{x}_T) + \big(\pi_T^k\big)^{\mathrm{T}}\boldsymbol{A}_T\boldsymbol{x}_T\\ &\qquad \geqslant \theta_T\big(\pi_T^k,\delta_T^k\big) - \big(\pi_T^k\big)^{\mathrm{T}}\boldsymbol{E}_{T-1}\boldsymbol{x}_{T-1}^j\\ &\qquad \forall k \in IU(T,j),\ \boldsymbol{x}_T \in \boldsymbol{R}^+.\end{aligned} \tag{20}$$

The new type of cut included in CG_{T-1}

$$\alpha_T(\boldsymbol{x}_{T-1}) + \sum_{k\in II(T,j)} \psi_T^{k,j}\big(\pi_T^k\big)^{\mathrm{T}}\boldsymbol{E}_{T-1}\boldsymbol{x}_{T-1} \geqslant \sum_{k\in II(T,j)} \psi_T^{k,j}\theta_T\big(\pi_T^k,\delta_T^k\big) \tag{21}$$

will be called type 2 Benders cuts.

Applying BT to partition the coordinator CG_{T-1} we obtain a new coordinator CG_{T-2}.

$$\begin{aligned}\mathrm{CG}_{T-2}:\quad & \min z = \sum_{t\in\Theta(T-2)} \big[\boldsymbol{c}_t^{\mathrm{T}}\boldsymbol{x}_t + \Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t)\big] + \alpha_{T-1}(\boldsymbol{x}_{T-2})\\ & \text{subject to}\quad \Omega_t(\boldsymbol{x}_{t-1},\boldsymbol{x}_t) + \big(\pi_t^k\big)^{\mathrm{T}}\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1} + \big(\pi_t^k\big)^{\mathrm{T}}\boldsymbol{A}_t\boldsymbol{x}_t\\ &\qquad \geqslant \theta_t\big(\pi_t^k,\delta_t^k\big)\quad \forall t\in\Theta(T-2)\quad \forall k\in IU,\\ &\qquad \boldsymbol{x}_t \in \boldsymbol{R}^+\quad \forall t\in\Theta(T-2),\\ &\qquad \alpha_{T-1}(\boldsymbol{x}_{T-2}) + \sum_{k\in II(T-1,j)} \psi_{T-1}^{k,j}\big(\pi_{T-1}^k\big)^{\mathrm{T}}\boldsymbol{E}_{T-2}\boldsymbol{x}_{T-2}\\ &\qquad \geqslant \sum_{k\in II(T-1,j)} \psi_{T-1}^{k,j}\theta_{T-1}\big(\pi_{T-1}^k,\delta_{T-1}^k\big)\\ &\qquad\quad + \sum_{m=1,IJ(T-1,j)} \gamma_{T-1}^{m,j} \sum_{k\in II(T,j)} \psi_T^{k,j}\theta_T\big(\pi_T^k,\delta_T^k\big)\\ &\qquad \forall j \in IJ(T-2),\end{aligned} \tag{22}$$

where $\gamma_t^{m,j}$ represents the dual variable of the mth type 2 Benders cut for period t obtained in the solution of the sub-problem $\mathrm{SG}_t(\boldsymbol{x}_{t-1}^j)$ defined for stage $T-1$ as

$$
\begin{aligned}
\mathrm{SG}_{T-1}\left(\boldsymbol{x}_{T-2}^{j}\right): \quad & \min \alpha_T\left(\boldsymbol{x}_{T-2}^{j}\right) = \boldsymbol{c}_{T-1}^{\mathrm{T}} \boldsymbol{x}_{T-1} + \Omega_{T-1}(\boldsymbol{x}_{T-2}, \boldsymbol{x}_{T-1}) + \alpha_T(\boldsymbol{x}_{T-1}) \\
& \text{subject to} \quad \Omega_{T-1}(\boldsymbol{x}_{T-2}, \boldsymbol{x}_{T-1}) + \left(\pi_{T-1}^{k}\right)^{\mathrm{T}} \boldsymbol{A}_{T-1} \boldsymbol{x}_{T-1} \\
& \qquad \geqslant \theta_T\left(\pi_{T-1}^{k}, \delta_{T-1}^{k}\right) - \left(\pi_{T-1}^{k}\right)^{\mathrm{T}} \boldsymbol{E}_{T-2} \boldsymbol{x}_{T-2}^{j} \\
& \qquad \forall k \in IU(T-1, j), \\
& \quad \boldsymbol{x}_{T-1} \in \boldsymbol{R}^{+}, \\
& \quad \alpha_T(\boldsymbol{x}_{T-1}) + \sum_{k \in II(T,j)} \psi_T^{k,j} \left(\pi_T^{k}\right)^{\mathrm{T}} \boldsymbol{E}_{T-1} \boldsymbol{x}_{T-1} \\
& \qquad \geqslant \sum_{k \in II(T,j)} \psi_T^{k,j} \theta_T\left(\pi_T^{k}, \delta_T^{k}\right) \quad \forall j \in IJ(T-1).
\end{aligned} \tag{23}
$$

It can be demonstrated that the coordinator sub-problem for each intermediate stage t (less than T) is

$$
\begin{aligned}
\mathrm{CG}_t: \quad & \min z = \sum_{\tau \in \Theta(t)} \left[\boldsymbol{c}_\tau^{\mathrm{T}} \boldsymbol{x}_\tau + \Omega_\tau(\boldsymbol{x}_{\tau-1}, \boldsymbol{x}_\tau)\right] + \alpha_{t+1}(\boldsymbol{x}_t) \\
& \text{subject to} \quad \Omega_\tau(\boldsymbol{x}_{\tau-1}, \boldsymbol{x}_\tau) + \left(\pi_\tau^{k}\right)^{\mathrm{T}} \boldsymbol{E}_{\tau-1} \boldsymbol{x}_{\tau-1} + \left(\pi_\tau^{k}\right)^{\mathrm{T}} \boldsymbol{A}_\tau \boldsymbol{x}_\tau \\
& \qquad \geqslant \theta_\tau\left(\pi_\tau^{k}, \delta_\tau^{k}\right) \quad \forall \tau \in \Theta(t)\ \forall k \in IU, \\
& \quad \boldsymbol{x}_\tau \in \boldsymbol{R}^{+} \quad \forall \tau \in \Theta(t), \\
& \quad \alpha_{t+1}(\boldsymbol{x}_t) + \sum_{k \in II(t+1,j)} \psi_{t+1}^{k,j} \left(\pi_{t+1}^{k}\right)^{\mathrm{T}} \boldsymbol{E}_t \boldsymbol{x}_t \geqslant \phi_t^{j} \\
& \qquad \forall j \in IJ(t)
\end{aligned} \tag{24}
$$

and the sub-problem for each intermediate stage t is

$$
\begin{aligned}
\mathrm{SG}_t\left(\boldsymbol{x}_{t-1}^{j}\right): \quad & \min \alpha_T\left(\boldsymbol{x}_{t-1}^{j}\right) = \boldsymbol{c}_t^{\mathrm{T}} \boldsymbol{x}_t + \Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t) + \alpha_{t+1}(\boldsymbol{x}_t) \\
& \text{subject to} \quad \Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t) + \left(\pi_t^{k}\right)^{\mathrm{T}} \boldsymbol{A}_t \boldsymbol{x}_t \\
& \qquad \geqslant \theta_T\left(\pi_t^{k}, \delta_t^{k}\right) - \left(\pi_t^{k}\right)^{\mathrm{T}} \boldsymbol{E}_{t-1} \boldsymbol{x}_{t-1}^{j} \quad \forall k \in IU(t, j), \\
& \quad \boldsymbol{x}_t \in \boldsymbol{R}^{+}, \\
& \quad \alpha_{t+1}(\boldsymbol{x}_t) + \sum_{k \in II(t+1,j)} \psi_{t+1}^{k,j} \left(\pi_{t+1}^{k}\right)^{\mathrm{T}} \boldsymbol{E}_t \boldsymbol{x}_t \geqslant \phi_t^{j} \\
& \qquad \forall j \in IJ(t),
\end{aligned} \tag{25}
$$

where ϕ_t^j is a constant value calculated as

$$
\phi_t^{j} = \begin{cases} \displaystyle\sum_{k \in II(T,j)} \psi_T^{k,j} \theta_T\left(\pi_T^{k}, \delta_T^{k}\right), & t = T, \\ \displaystyle\sum_{k \in II(t,j)} \psi_t^{k,j} \theta_t\left(\pi_t^{k}, \delta_t^{k}\right) + \sum_{m=1, IJ(t,j)} \gamma_t^{m,j} \phi_{t+1}^{m}, & t = 1, T-1. \end{cases} \tag{26}
$$

The original problem GDP corresponds to CG_1, the coordinator for the first stage. The functions $\Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$ corresponds to the supply functions that Read considers in the DDP-R approach.

4. Special cases

4.1. $\boldsymbol{B}_t$, $\boldsymbol{G}_t$ and $\boldsymbol{d}_t$ time independent

Now we consider the special case when the dual feasibility zone of the problems $\mathrm{SU}_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$ is static, which implies that the matrices $\boldsymbol{B}_t$ and $\boldsymbol{G}_t$ and the vector $\boldsymbol{d}_t$ are time independent; then, the dual problem is:

$$\begin{aligned} \mathrm{DSU}_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t): \quad & \max \Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t) = \pi_t^{\mathrm{T}}[\boldsymbol{b}_t - \boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1} - \boldsymbol{A}_t\boldsymbol{x}_t] + \delta_t^{\mathrm{T}}\boldsymbol{g}_t \\ & \text{subject to} \quad \pi_t^{\mathrm{T}}B + \delta_t^{\mathrm{T}}G \leqslant d^{\mathrm{T}}. \end{aligned} \tag{27}$$

In this case, a feasible vector of dual variables $\{\pi_t, \delta_t\}$ for any $\mathrm{DSU}_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$ is feasible for all $\mathrm{DSU}_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$ independent of the value of t; when we solve $\mathrm{SU}_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$ for a specific value of t we can generate type 1 Benders cuts for all periods and the coordinator CX can be expressed as

$$\begin{aligned} \mathrm{CX}: \quad & \min \sum_{t\in\Theta(T)} \boldsymbol{c}_t^{\mathrm{T}}\boldsymbol{x}_t + \Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t) \\ & \text{subject to} \quad \Omega_t(\boldsymbol{x}_{t-1,}\boldsymbol{x}_t) + \left(\pi^k\right)^{\mathrm{T}}\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1} + \left(\pi^k\right)^{\mathrm{T}}\boldsymbol{A}_t\boldsymbol{x}_t \\ & \qquad\qquad \geqslant \theta_t\left(\pi^k, \delta^k\right) \quad \forall t \in \Theta(T)\ \forall k \in IT, \end{aligned} \tag{28}$$

where the dual variables vector $\{\pi, \delta\}$ is time independent; then, for each iteration of the coordinator-sub-problems we need to solve only one problem $\mathrm{SU}_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$.

This situation is very common for matrices $\boldsymbol{G}_t$ and $\boldsymbol{B}_t$ because they are related with the technology and with the topology of the modeled system, which for the short term, is normally time independent. The vector $\boldsymbol{d}_t$ is related with the costs and, in many cases, it is time dependent, but often the time variation of $\boldsymbol{d}_t$ is expressed as

$$\boldsymbol{d}_t = \beta^t \boldsymbol{d}, \tag{29}$$

where β is a discount factor and $\boldsymbol{d}$ is a constant vector reference price. In this case, we can express the sub-problems as

$$\begin{aligned} \mathrm{SU}_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t): \quad & \min \Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t) = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{u}_t \\ & \text{subject to} \quad \boldsymbol{B}\boldsymbol{u}_t = \boldsymbol{b}_t - \boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1} - \boldsymbol{A}_t\boldsymbol{x}_t, \boldsymbol{G}\boldsymbol{u}_t = \boldsymbol{g}_t, \boldsymbol{u}_t \in \boldsymbol{R}^+ \end{aligned} \tag{30}$$

and the coordinator CX as

$$\begin{aligned} \mathrm{CX}: \quad & \min \sum_{t\in\Theta(T)} \boldsymbol{c}_t^{\mathrm{T}}\boldsymbol{x}_t + \beta^t\Omega_t(\boldsymbol{x}_{t-1,}\boldsymbol{x}_t) \\ & \text{subject to} \quad \Omega_t(\boldsymbol{x}_{t-1,}\boldsymbol{x}_t) + \left(\pi^k\right)^{\mathrm{T}}\boldsymbol{E}_{t-1}\boldsymbol{x}_{t-1} + (\pi^k)^{\mathrm{T}}\boldsymbol{A}_t\boldsymbol{x}_t \\ & \qquad\qquad \geqslant \theta_t\left(\pi^k, \delta^k\right) \quad \forall t \in \Theta(T)\ \forall k \in IT \end{aligned} \tag{31}$$

and we need to solve only one problem in the sub-problem level, independently of the numbers of periods. This fact is very important when the number periods is very large, such as in the case of the control problems, that require many short periods to represent adequately continuous movement and constraints of the state variables.

From an economic point of view, $\Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$ represents the optimum operation costs in the period t as a consequence of the boundary conditions ($\boldsymbol{x}_{t-1}$ and $\boldsymbol{x}_t$). Frequently, $\Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$ is time independent in the short and the medium term planning horizons, due to the fact that in the planning horizon the technology, the topology and the price index do not change.

Other special cases can be considered when the vector reference price $\boldsymbol{d}$ is time dependent and has seasonal variations. In this case we can define families of functions $\Omega_t^e(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$, where the index e represents the seasonal variation of $\boldsymbol{d}$, that is $\boldsymbol{d}^e$. Each period t belongs to a "season" and we must solve sub-problems for each type of season.

4.2. $\boldsymbol{A}_t = \boldsymbol{I}$ *and* $\boldsymbol{E}_t = -\boldsymbol{I}$

Another special case may be considered when $\boldsymbol{A}_t$ is equal to the identity matrix and $\boldsymbol{E}_t$ to the negative identity matrix, then the definition of $\boldsymbol{x}_t$ is expressed as

$$\boldsymbol{x}_t = \boldsymbol{x}_{t-1} + \boldsymbol{b}_t - \boldsymbol{B}_t \boldsymbol{u}_t \quad \forall t \in \Theta(T). \tag{32}$$

This is a very common case for stock equations, in which the storage level $\boldsymbol{x}_t$ is a function of the storage level in $t-1$ plus an external input and plus a linear combination of the production and the distribution variables.

Under this condition the function $\Omega_t(\boldsymbol{x}_{t-1,}\boldsymbol{x}_t)$ can be stated as $\Omega_t(\Delta \boldsymbol{x}_t)$ since the sub-problem may be expressed as

$$\begin{aligned} \mathrm{SU}_t(\Delta \boldsymbol{x}_t): \quad & \min \Omega_t(\Delta \boldsymbol{x}_t) = \boldsymbol{d}_t^{\mathrm{T}} \boldsymbol{u}_t \\ & \text{subject to} \quad \boldsymbol{B}_t \boldsymbol{u}_t = \boldsymbol{b}_t - \Delta \boldsymbol{x}_t, \boldsymbol{G}_t \boldsymbol{u}_t = \boldsymbol{g}_t, \boldsymbol{u}_t \in \boldsymbol{R}^+. \end{aligned} \tag{33}$$

The advantage of this structure is that the optimum operation cost function $\Omega_t(\boldsymbol{x}_{t-1}, \boldsymbol{x}_t)$ only depends on the variation of the state variables $\Delta \boldsymbol{x}_t$, but is independent of its absolute level. The coordinator CX may be expressed in terms of $\Delta \boldsymbol{x}_t$ as

$$\begin{aligned} \mathrm{CX}: \quad & \min \sum_{t \in \Theta(T)} \boldsymbol{c}_t^{\mathrm{T}} \sum_{\tau=1,t} \Delta \boldsymbol{x}_\tau + \beta^t \Omega_t(\Delta \boldsymbol{x}_t) \\ & \text{subject to} \quad \Omega_t(\Delta \boldsymbol{x}_t) + \left(\pi^k\right)^{\mathrm{T}} \Delta \boldsymbol{x}_t \geqslant \theta_t\left(\pi^k, \delta^k\right) \quad \forall t \in \Theta(T)\ \forall k \in IT. \end{aligned} \tag{34}$$

5. Conclusions

The conceptual formulation of the GDDP problem enables development of efficient algorithms based on the partition and the decomposition of the original problem using Benders' theory. The solution of the original problem is found by the coordinated solution of multiple problems of smaller dimension. In some cases, it is possible to visualize special matrix structures to generate Benders' cuts for all periods and eliminates the need to solve a problem for each period. This is of special importance when the number of periods is very large, as in the case of optimal control problems.

The next step in the development of the GDDP theory is its extension to include stochastic optimization problems.

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