2016 School Year

Graduate School
Entrance Examination Problem Booklet

Mathematics

Examination Time: 10:00 to 12:30

Instructions

1. Do not open this problem booklet until the start of the examination is announced.
2. If you find missing, misplaced, and/or unclearly printed pages in the problem booklet, ask the examiner.
3. Answer all of three problems appearing in this booklet, in Japanese or English.
4. You are given three answer sheets. You must use a separate answer sheet for each problem. You may continue to write your answer on the back of the answer sheet if you cannot conclude it on the front.
5. Fill the designated blanks at the top of each answer sheet with your examinee's number and the problem number you are to answer.
6. The blank pages are provided for rough work. Do not detach them from this problem booklet.
7. An answer sheet is regarded as invalid if you write marks and/or symbols and/or words unrelated to the answer on it.
8. Do not take either the answer sheets or the problem booklet out of the examination room.

Examinee's number	No.

Fill this box with your examinee's number.

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Problem 1

The tribonacci numbers $\{T_n\}$ are defined for non-negative integers n as follows.

$$\begin{cases} T_0 = T_1 = 0 \\ T_2 = 1 \\ T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad (n \geq 0). \end{cases}$$

Answer the following questions.

(1) Find the matrix A that satisfies Eq. (1.1) for all non-negative integers n.

$$\begin{pmatrix} T_{n+3} \\ T_{n+2} \\ T_{n+1} \end{pmatrix} = A \begin{pmatrix} T_{n+2} \\ T_{n+1} \\ T_n \end{pmatrix}. \tag{1.1}$$

(2) Find the rank and the characteristic equation, i.e., the equation that eigenvalues satisfy, of the matrix A.

(3) Let $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of the matrix A. Express an eigenvector corresponding to each of the eigenvalues using $\lambda_1, \lambda_2, \lambda_3$.

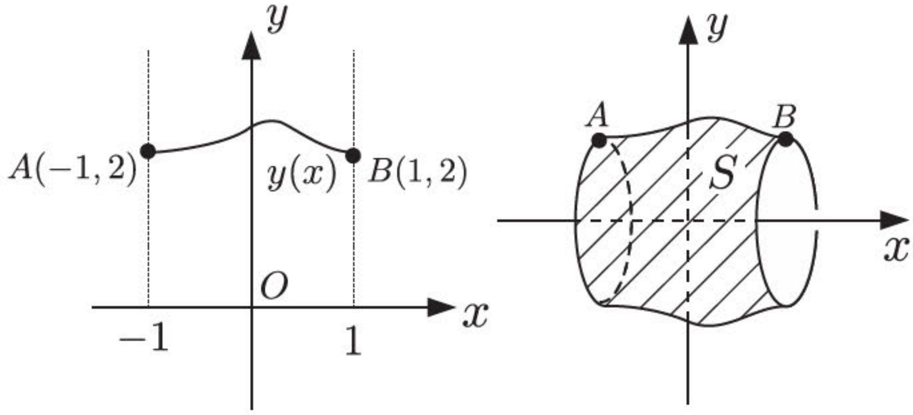
(4) Prove that the matrix A has only one real number eigenvalue. Letting λ_1 correspond to this eigenvalue, prove that $1 < \lambda_1 < 2$.

(5) Prove that T_n can be expressed as $T_n = c_1\lambda_1^n + c_2\lambda_2^n + c_3\lambda_3^n$ using constant complex numbers c_1, c_2, c_3. You do not need to find values of c_1, c_2, c_3 explicitly.

(6) Prove $\displaystyle\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = \lambda_1$.

Problem 2

Consider a twice differentiable function $y(x)$ in an xy plane which connects two points $A(-1, 2)$ and $B(1, 2)$. Let S be outer surface area of the cylindrical object created by rotation of the curve $y(x)$ about the x axis. Answer the following questions.



(1) Prove that the surface area S is given by

$$S = 2\pi \int_{-1}^{1} F(y, y')\mathrm{d}x, \tag{2.1}$$

$$F(y, y') = y\sqrt{1 + (y')^2}, \tag{2.2}$$

where $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(2) Let the curve $y(x)$ satisfy the following Euler–Lagrange equation for arbitrary x:

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0. \tag{2.3}$$

Considering Eq. (2.3) along with $\dfrac{\mathrm{d}F}{\mathrm{d}x}$, prove that the following relation holds:

$$F - y'\frac{\partial F}{\partial y'} = c. \tag{2.4}$$

Here c is a constant.

(3) Express a differential equation satisfied by the curve $y(x)$ using y, y', c.

(4) Represent the curve $y(x)$ as a function of x and c.
Obtain an equation which should be satisfied by the constant c.

Problem 3

Answer the following questions.

(1) Calculate the number of possible ways to distribute n equivalent balls to r distinguishable boxes such that each box contains at least one ball, where $n \geq 1$ and $1 \leq r \leq n$.

Next, consider to place n black balls and m white balls in a line uniformly at random. A run is defined to be a succession of the same color. Let r be the number of runs of black balls and s be the number of runs of white balls. Assume that $n \geq 1$, $m \geq 1$, $1 \leq r \leq n$, and $1 \leq s \leq m$. For example, the sequence

$$\underline{\bullet\,\bullet}\;\underline{\circ\,\circ\,\circ}\;\underline{\bullet\,\bullet\,\bullet}\;\underline{\circ\,\circ}\;\underline{\bullet}$$

corresponds to $r = 3$ and $s = 2$.

(2) Calculate the total number of arrangements when we do not distinguish among balls of the same color.

(3) Calculate the probability $P(r, s)$ that the number of runs of black balls is r and the number of runs of white balls is s.

(4) Calculate the probability $P(r)$ that the number of runs of black balls is r.

(5) Using $(1 + x)^n(1 + x)^m = (1 + x)^{n+m}$, show that the following equations hold.

$$\sum_{\ell=0}^{\min\{n,m\}} \binom{n}{\ell}\binom{m}{\ell} = \binom{n+m}{m} \tag{3.1}$$

$$\sum_{\ell=0}^{\min\{n-1,m\}} \binom{n}{\ell+1}\binom{m}{\ell} = \binom{n+m}{m+1} \tag{3.2}$$

(6) Calculate the expected value $E(r)$ and the variance $V(r)$ of r.
Calculate $\lim_{N\to\infty} \frac{E(r)}{N}$ and $\lim_{N\to\infty} \frac{V(r)}{N}$ supposing that $N = n+m$ and $\lim_{N\to\infty} \frac{n}{N} = \lambda$, where λ is a real constant.

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