105年 台灣大學 試題

解題老師：周易

系所：生物環境工程所(甲組)　　科目：工程數學(I)

1. Find the general solution. (10%)

$y' + y\sin x = e^{\cos x}$

2. Find the particular solution. (15%)

$x^2 y'' - 4xy' + 6y = 0$, $y(1) = 0.4$, $y'(1) = 0$

3. Find the Laplace Transform. Show the details. (10%)

(1) $(a - bt)^2$

(2) $e^{3t}\sinh t$

4. Given an ODE $y'' + 3y' + 2y = 10\sin t$. Find the general solution by Laplace transform. (15%)

5. Solve the heat equation by Fourier series. Assume there is a laterally insulated bar with temperature 0 at both end; the initial temperature of the bar is (25%)

$f(x) = x \quad \text{if} \quad 0 < x < 5$

$f(x) = (10 - x) \quad \text{if} \quad 5 < x < 10$

Derive two ordinary differential equations (ODEs) from the heat equation. Then solve ODEs with boundary and initial conditions. Show the final solution of the problem.

6. Determine eigenvalues and eigenvectors of the following matrix. (25%)

$$A = \begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix}$$

An elastic membrane with boundary $x_1^2 + x_2^2 = 1$ is stretched by the following equation such that (x_1, x_2) goes to (y_1, y_2) . Plot the original boundary, new boundary, and principal stretch directions.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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1. $dy + y\sin x dx = e^{\cos x}\cdot dx$， $I = \exp\left[\int \sin x dx\right] = e^{-\cos x}$

$e^{-\cos x}[dy + y\sin x dx] = e^{-\cos x}\cdot e^{\cos x}dx$， $d[ye^{-\cos x}] = dx$

$ye^{-\cos x} = x + c$， $y = (x+c)e^{\cos x}$ 爲解

2. 令 $x = e^t$， $t = \ln x$ 代入 $x^2y'' - 4xy' + 6y = 0$， $D_t(D_t - 1)y - 4D_t y + 6y = 0$

$(D_t^2 - 5D_t + 6)y = 0$， $(D_t - 2)(D_t - 3)y = 0$， $y = c_1e^{2t} + c_2e^{3t} = c_1x^2 + c_2x^3$

代入條件 $y(1) = 0.4$， $c_1 + c_2 = 0.4$， $c_1 = 1.2$， $c_2 = -0.8$

代入 $y'(1) = 0$， $2c_1 + 3c_2 = 0$

$y = 1.2x^2 - 0.8x^3$

3. (1) $L[(a-bt)^2] = L[a^2 - 2abt + t^2\cdot b^2] = \dfrac{a^2}{s} - \dfrac{2ab}{s^2} + \dfrac{2b^2}{s^3}$

(2) $L[e^{3t}\sinh t] = L[\sinh t]_{s=s-3} = \left.\dfrac{1}{s^2-1}\right|_{s=s-3} = \dfrac{1}{(s-3)^2-1} = \dfrac{1}{s^2-6s+8}$

4. $L[y'' + 3y' + 2y] = 10\cdot L[\sin t]$，設 $y(0) = c_1$， $y'(0) = c_2$

$(s^2Y - sc_1 - c_2) + 3(sY - c_1) + 2Y = \dfrac{10}{s^2+1}$， $(s^2 + 3s + 2)Y = sc_1 + c_2 + 3c_1 + \dfrac{10}{s^2+1}$

$$Y = \frac{sc_1 + c_2 + 3c_1}{(s+1)(s+2)} + \frac{10}{(s+1)(s+2)(s^2+1)}$$

$$Y = \frac{sc_1 + c_2 + 3c_1}{(s+1)(s+2)} + \frac{5}{s+1} + \frac{-2}{s+2} + \frac{-3s+1}{s^2+1}$$

$Y = \dfrac{k_1}{s+1} + \dfrac{k_2}{s+2} + \dfrac{-3s+1}{s^2+1}$， $y = L^{-1}[Y] = k_1e^{-t} + k_2e^{-2t} - 3\cos t + \sin t$

5. heat equation $\dfrac{\partial u}{\partial t} = \alpha\dfrac{\partial^2 u}{\partial x^2}$， $0 < x < 10$， $t > 0$， α 爲常數

$u(0,t) = 0$， $u(10,t) = 0$， $u(x,0) = f(x)$

令 $u(x,t) = X(x)\cdot T(t)$ 代入 PDE， $XT' = \alpha X''T$， $\dfrac{T'}{\alpha T} = \dfrac{X''}{X} = \lambda$

$X'' - \lambda X = 0$， $T' - \lambda\alpha T = 0$， $X(0) = 0$， $X(10) = 0$

當 $\lambda > 0$，令 $\lambda = \beta^2$， $\beta > 0$， $X = c_1 \cosh \beta x + c_2 \sinh \beta x$

代入 $X(0) = 0$， $c_1 = 0$， $X(10) = 0$， $c_2 \sinh 10\beta = 0$， $c_2 = 0$

當 $\lambda = 0$， $X = c_1 + c_2 x$，代入 $X(0) = 0$， $c_1 = 0$， $X(10) = 0$， $c_2 = 0$

當 $\lambda < 0$，令 $\lambda = -\beta^2$， $\beta > 0$， $X = c_1 \cos \beta x + c_2 \sin \beta x$

代入 $X(0) = 0$， $c_1 = 0$，代入 $X(10) = 0$， $c_2 \sin 10\beta = 0$

當 $10\beta = n\pi$， $n = 1, 2, 3, \cdots$， c_2 為任意數

特徵值 $\lambda_n = -\beta^2 = -\left(\dfrac{n\pi}{10}\right)^2$，特徵值數 $X_n(x) = c_n \sin\dfrac{n\pi x}{10}$

將 $\lambda_n = -\dfrac{n^2\pi^2}{100}$ 代入 $T' - \lambda\alpha T = 0$， $T' + \dfrac{n^2\pi^2}{100}\alpha T = 0$

$T = k\exp\left(-\dfrac{n^2\pi^2}{100}\alpha t\right)$，代入 $u(x,t) = X(x)\cdot T(t)$

$u = c_n \sin\dfrac{n\pi x}{10}\cdot k\exp\left(-\dfrac{\alpha n^2\pi^2}{100}t\right)$，令 $B_n = C_n \cdot k$

$u = B_n \cdot \exp\left(-\dfrac{\alpha n^2\pi^2}{100}t\right)\cdot\sin\dfrac{n\pi x}{10}$， $n = 1, 2, 3, \cdots$

都是齊性 PDE 之齊性解, 齊性 PDE 通解

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cdot \exp\left(-\frac{\alpha n^2\pi^2}{100}t\right)\cdot\sin\frac{n\pi x}{10}$$

代入 $t = 0$， $u(x,0) = f(x)$， $f(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\dfrac{n\pi x}{10}$

$$B_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}dx = \frac{1}{5}\left[\int_0^5 x\sin\frac{n\pi x}{10}dx + \int_5^{10}(10-x)\sin\frac{n\pi x}{10}dx\right]$$

$$B_n = \frac{1}{5}\left[-\frac{10x}{n\pi}\cos\frac{n\pi x}{10}\bigg|_0^5 + \frac{10}{n\pi}\int_0^5 \cos\frac{n\pi x}{10}dx - \frac{10}{n\pi}(10-x)\cos\frac{n\pi x}{10}\bigg|_5^{10}\right.$$
$$\left.-\frac{10}{n\pi}\int_5^{10}\cos\frac{n\pi x}{10}dx\right]$$

$$B_n = \frac{1}{5}\left[-\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$u = \displaystyle\sum_{n=1}^{\infty}\frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}\cdot\exp\left(-\frac{\alpha n^2\pi^2}{100}t\right)\cdot\sin\frac{n\pi x}{10}$ 為解

6. $A\vec{x}=\lambda\vec{x}$，$(A-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix}\frac{5}{2}-\lambda & \frac{1}{2}\\ \frac{1}{2} & \frac{5}{2}-\lambda\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\vec{0}\text{，}\left|A-\lambda I\right|=(\lambda-2)(\lambda-3)$$

特徵值 $\lambda=2,3$

當 $\lambda_1=2$, eigenvectors $\vec{u}=c_1\begin{bmatrix}1\\-1\end{bmatrix}$, $\lambda_2=3$，$\vec{u}=c_2\begin{bmatrix}1\\1\end{bmatrix}$

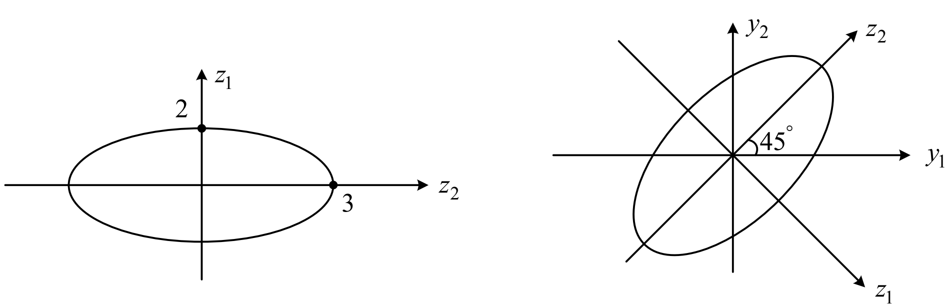
取 $S=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$, $D=\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$, $S^{-1}=S^T$，$A=SDS^{-1}$爲正交對角化

$\vec{y}=A\vec{x}$，$\vec{x}=A^{-1}\vec{y}$, 代入限制條件$\left\|\vec{x}\right\|^2=\vec{x}^T\vec{x}=1$

$(A^{-1}\vec{y})^T(A^{-1}\vec{y})=1$，$\vec{y}^T(A^{-1})^TA^{-1}\vec{y}=1$，$\vec{y}^T(A^T)^{-1}A^{-1}\vec{y}=1$

$\vec{y}^TA^{-1}A^{-1}\vec{y}=1$，$\vec{y}^TSD^{-2}S^{-1}\vec{y}=1$，$(S^T\vec{y})^T\cdot D^{-2}\cdot(S^T\vec{y})=1$

令 $\vec{z}=S^T\vec{y}$，$\vec{z}^T\begin{bmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{9}\end{bmatrix}\vec{z}=1$，$\frac{1}{4}z_1^2+\frac{1}{9}z_2^2=1$



$\because\ \vec{z}=S^T\vec{y}=S^{-1}\vec{y}$, $\begin{bmatrix}z_1\\z_2\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$

$$\begin{cases}z_1=\frac{1}{\sqrt{2}}y_1-\frac{1}{\sqrt{2}}y_2\\ z_2=\frac{1}{\sqrt{2}}y_1+\frac{1}{\sqrt{2}}y_2\end{cases},\quad \begin{cases}\vec{e_1}=\frac{1}{\sqrt{2}}\vec{i}-\frac{1}{\sqrt{2}}\vec{j}\\ \vec{e_2}=\frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{j}\end{cases}$$

$\vec{e_1}$ 與 $\vec{e_2}$ 爲 principal direction