TROUBLESOME FRACTIONS

James Tanton

CONTENTS:

WHAT IS A FRACTION?

As we shall see answering this question is far from easy! It is incredibly hard to pin down exactly what a fraction is. (If you want to see the full details right now as to why fractions are so philosophically thorny, take a peak at the essay at the end of this pamphlet on page 63.)

As one goes through the early grades of school one is introduced to different ideas as to what a fraction could be – a portion of pie, an answer to a sharing problem, a point on a number line, an actual number in its own right, and more – and it is far from obvious if all these different attempts to answer what a fraction is are the same. One goes from grade-to-grade with different pictures in mind and it is usually not clear if you are meant to forget your understanding of fractions from previous grades or not. It is all very, very confusing.

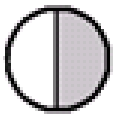
In the next number of pages we'll explore different attempts to answer the question as to what a fraction is. But this feat is akin to the ancient Indian parable of blind men feeling an elephant. Each speaks a truth: "An elephant is a flat expanse of leather" (feeling its belly). "An elephant is a hard bone" (feeling the tusk). "An elephant is a length of rope" (feeling its tail). But no answer actually says what an elephant is in totality. We've give students different aspects of a truth of what a fraction is and over the course of the early grades these answers feel contradictory. No wonder we are all left unsettled and uncomfortable with fractions and view them with deep suspicion!

In these notes we'll go through some possible answers to what a fraction is. Then we'll see if we can pull out the features of each answer that seem to point to truth, to key properties we like to believe about these things.

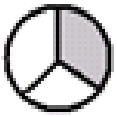
That is, we'll go through the various stories of fractions, and then let them go! We'll let the mathematics itself be the ultimate story that explains everything we hold true about fractions.

MODEL 1:
A FRACTION IS A PART OF A WHOLE

In the early grades we usually model fractions as parts of a whole. And pie seems to be the favoured whole. We draw pictures of half a pie:

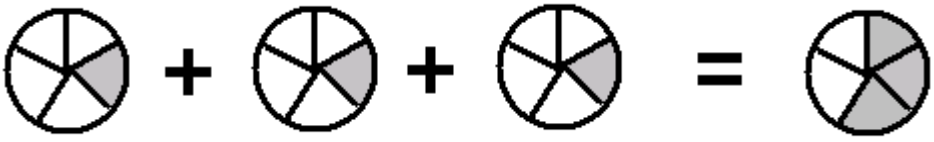
$\frac{1}{2} =$ 

and a third of a pie:

$\frac{1}{3} =$ 

and so on.

We can take three fifths, and draw a picture of it,  , and we are taught to write this as $\frac{3}{5}$. This is actually a bit confusing! Here what three fifths actually look like:



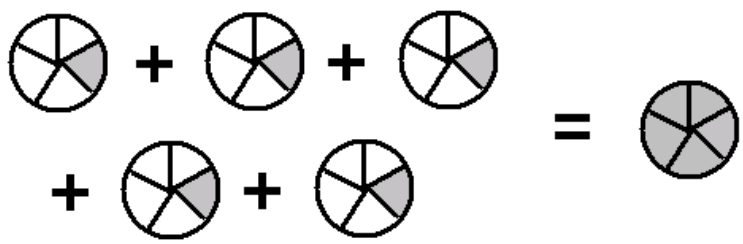
So three fifths is $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$. This is three groups of one-fifth, which, in terms of multiplication is $3\times\frac{1}{5}$:

$$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=3\times\frac{1}{5}$$

I'M CONFUSED! Should I wonder if $3\times\frac{1}{5}$ and $\frac{3}{5}$ are the same thing or not? Is this even a question? Maybe "$\frac{3}{5}$" is a shorthand for $3\times\frac{1}{5}$, three groups of one-fifth?

It is usually assumed, without explicit mention, that $\frac{a}{b}$ is indeed shorthand for a groups of $\frac{1}{b}$.

Given this shorthand …What is $\frac{5}{5}$? It's $5 \times \frac{1}{5}$, five groups of one fifth:



That's one whole. So we have: $\frac{5}{5} = 1$.

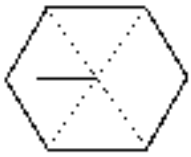
EXERCISE 1: What is $\frac{2}{2}$ shorthand for? Draw a picture to show, in this thinking, $\frac{2}{2} = 1$. Draw a picture to explain why, in this thinking, $\frac{4}{2} = 2$.

To summarize:

> **If b is a positive whole number and we divide a pie into b equal parts, then one of those slices is denoted $\frac{1}{b}$.**
>
> **If a is a positive whole number, then $\frac{a}{b}$ is shorthand for $a \times \frac{1}{b}$, that is, a of those slices.**

So here, in this model, <u>a fraction is an actual amount of pie</u>. We have that $\frac{a}{b}$ is a slices of actual pie.

EXERCISE 2: There is nothing special about circular pies. For example, a hexagonal pie is good for illustrating the fractions $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$ and $\frac{6}{6}$.



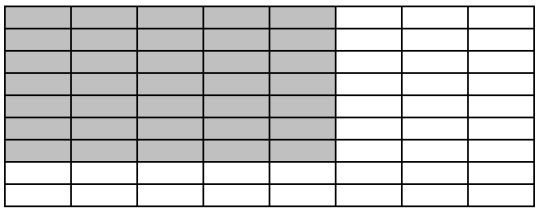
a) Why is this shape used? What does the amount of pie associated with the fraction $\frac{1}{6}$ look like?

b) What does the amount of pie associated with the fraction $\frac{6}{6}$ look like?

c) What shape pie would be good for illustrating the fractions $\frac{1}{8}$ up to $\frac{8}{8}$?

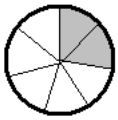
d) Are $\frac{4}{8}$ and $\frac{1}{2}$ the same fraction? Explain.

EXERCISE 3: What fraction is represented by the shaded portion of this rectangular pie?

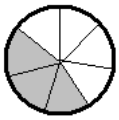


ADDING EASY FRACTIONS IS EASY IN THIS MODEL

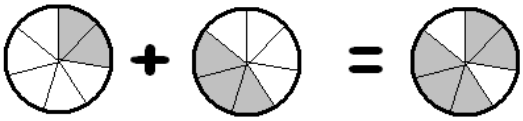
The fraction $\frac{2}{7}$ is two sevenths of a pie.



The fraction $\frac{3}{7}$ is three sevenths of a pie.



And adding pieces of pie together makes good practical sense in this pie model:



We see that the answer is $\frac{5}{7}$:

$$\frac{2}{7}+\frac{3}{7}=\frac{5}{7}.$$

Most people just read this as "Two sevenths plus three sevenths gives five sevenths" and think that the problem is just as easy as saying "two apples plus three apples gives five apples." And in this model it is! We are adding together objects in units of sevenths.

In this model we like to believe:

$$\frac{a}{N}+\frac{b}{N}=\frac{a+b}{N}$$

(Here a, b, and N are positive whole numbers.)

Adding fractions given in the same units is easy. If we mix the units, however, matters get thorny. For example, I feel like I can draw a picture for $\frac{2}{3}+\frac{5}{19}$ but I am not sure if I want to write the answer as a single fraction. That seems hard.

MODEL 2:
A FRACTION IS AN ANSWER TO A DIVISION PROBLEM

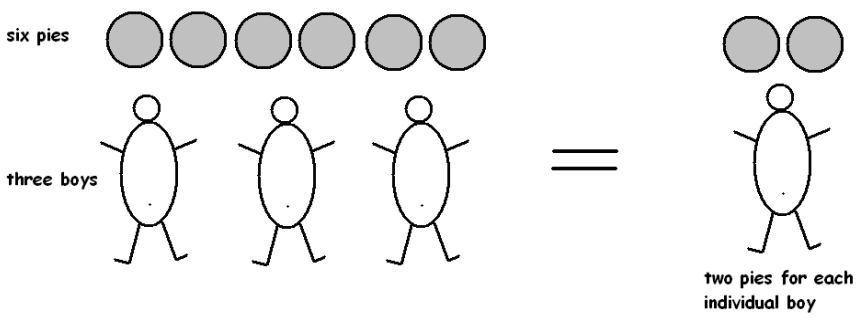
When we divide a pie into equal parts - as for the previous model - we are really sharing the pie equally among a group of people. All the ideas from the previous models are the results of sharing. So really …

MODEL 2: **A fraction is an answer to a division problem.**

For example, suppose 6 pies are to be shared equally among 3 boys. This yields 2 pies per boy. We write:

$$\frac{6}{3} = 2.$$

(We could, of course, also write $6 \div 3 = 2$ or $\begin{array}{r} 2 \\ 3\overline{)6} \end{array}$.)



Here the fraction "$\frac{6}{3}$", our division problem, is equivalent to the number 2. It represents the number of pies one whole boy receives.

In the same way …

sharing 10 pies among 2 boys yields: $\frac{10}{2} = 5$ pies per boy.

sharing 8 pies among 2 boys yields: $\frac{8}{2} = 4$

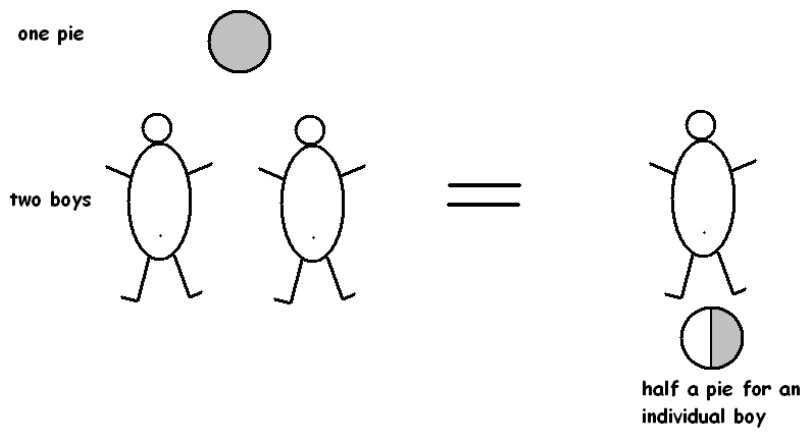
sharing 5 pies among 5 boys yields: $\frac{5}{5}=1$

and

the answer to sharing 1 pie among 2 boys is $\frac{1}{2}$, which we call one half.

This final example shows that our new answer to what a fraction is really is a better answer than model 1. The pictures were we drawing in model 1 really were the results of sharing problems.

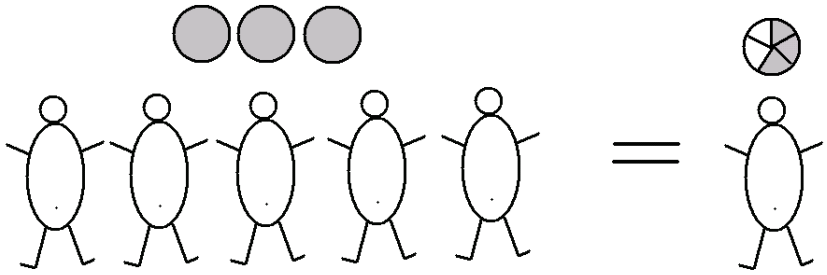
If one pie is shared (equally) between two boys, then each boy receives a portion of a pie which we choose to call "half."



In the same way, the picture  represents "one third," that is, $\frac{1}{3}$, the amount of pie an individual boy would receive if one pie is shared among three.

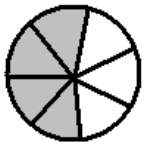
The picture  is $\frac{1}{5}$, the amount of pie an individual boy receives when one pie is shared among five.

Think about the fraction $\frac{3}{5}$. In our new sharing model this is the answer to sharing three pies equally among five boys:



We could accomplish this by dividing each pie into five equal parts and giving each boy three of those parts.

EXERCISE 4: Here is the answer to a division problem:



This represents the amount of pie an individual boy receives if some number of pies is shared among some number of boys.

How many pies? _____
How many boys? _____

EXERICSE 5: Leigh says that "$\frac{3}{5}$ is three times as big as $\frac{1}{5}$." She argues:

In one room, three pies are shared among five boys.
In another room, one pie is shares among five boys.

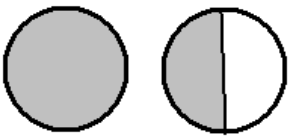
Is it clear that each boy in the first room receives three times as much pie as each boy in the second room? Is this a valid way to think about matters?

EXERCISE 6: What does the division problem $\frac{1}{1}$ represent? How much pie does an individual boy receive?

EXERCISE 7: What does the division problem $\frac{5}{1}$ represent? How much pie does an individual boy receive?

EXERCISE 8: What does the division problem $\frac{8}{8}$ represent? How much pie does an individual boy receive?

EXERCISE 9: A CHALLENGE
Here is the answer to another division problem. This is the amount of pie an individual boy receives

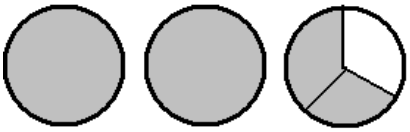


How many pies were there in the division problem? _____

How many boys were there in the division problem? _____

Are you clear that your answers are correct?

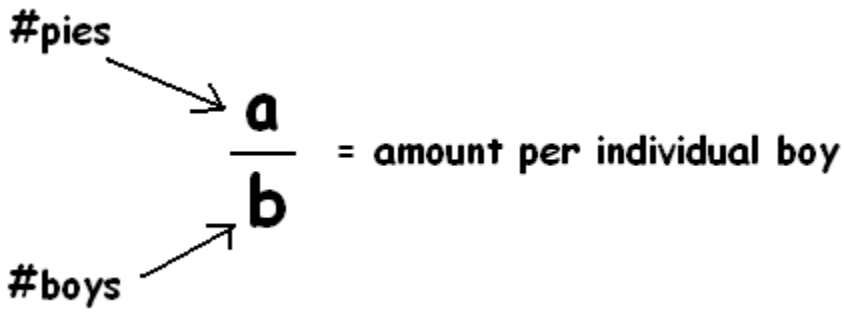
EXERCISE 10: MORE OF A CHALLENGE:
How many pies and how many boys for this answer?



Number of pies: _____
Number of boys: _____

In this second model:

A fraction $\frac{a}{b}$ represents the amount of pie an individual boy receives when a pies are shared equally among b boys.



(We are assuming, for now, that both a and b are positive numbers.)

Notice now that fractions are now quantities in terms of <u>pie per boy</u>. Something has indeed subtly changed in going from model 1 to model 2. (In model 1, fractions were just pie.)

EXERCISE 11: What is $\frac{2}{2}$? $\frac{7}{7}$? $\frac{100}{100}$? What is $\frac{a}{a}$ for any positive whole number a ?

EXERCISE 12: What is $\frac{1876}{1}$?

Exercises 11 and 12 suggest that for each positive whole number a we have:

$$\frac{a}{a} = 1$$

and

$$\frac{a}{1} = a \, .$$

EXERCISE 13: "I have no pies to share among seven boys." Use this to make a statement about a division problem and hence a statement about fractions.

PLAYING WITH NUMERATORS AND DENOMINATORS:

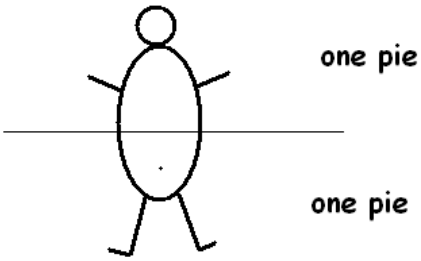
For a fraction $\frac{a}{b}$, the top number a (which, for us, is the number of pies) is called the <u>numerator</u> of the fraction, and the bottom number b (the number of pies), the <u>denominator</u> of the fraction. Most people insist that these numbers each be whole numbers, but they really don't have to be.

To see what I mean, let's have some fun!

QUESTION: What does $\frac{1}{\left(\frac{1}{2}\right)}$ represent?

This means assigning one pie to each "group" of half a boy. So how much would a whole boy receive?

Answer: Two pies!



We have:

$$\frac{1}{\left(\frac{1}{2}\right)} = 2$$

COMMENT: It seems we have returned to model 1 here. What do we mean by "half a boy"? Or maybe we are thinking gruesome thoughts a la model 2 and are sharing a boy equally among two ogres!

QUESTION: What does $\frac{1}{\left(\frac{1}{3}\right)}$ represent?

Answer: Distributing one pie to each third of a boy yields the result of 3 whole pies for an individual boy.

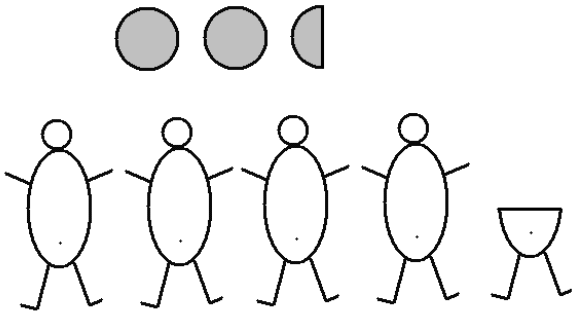
$$\frac{1}{\left(\frac{1}{3}\right)} = 3$$

EXERCISE 14: What is the answer to $\frac{1}{\left(\frac{1}{6}\right)}$?

EXERCISE 15: The answer to $\frac{5}{\left(\frac{1}{2}\right)}$ is ten. Do you we why? [How many pies are given to half a boy?]

EXERCISE 16: What is the answer to $\frac{4}{\left(\frac{1}{3}\right)}$?

SCARY COMPLETELY-OPTIONAL CHALLENGE:
Two-and-a-half pies are to be shared equally among four-and-a-half boys!



How much pie does an individual (whole) boy receive?

This is a very tricky problem. Only attempt this if it seems fun to do so. We'll see a very easy way to think about these types of problems a little bit later.

THE KEY FRACTION RULE

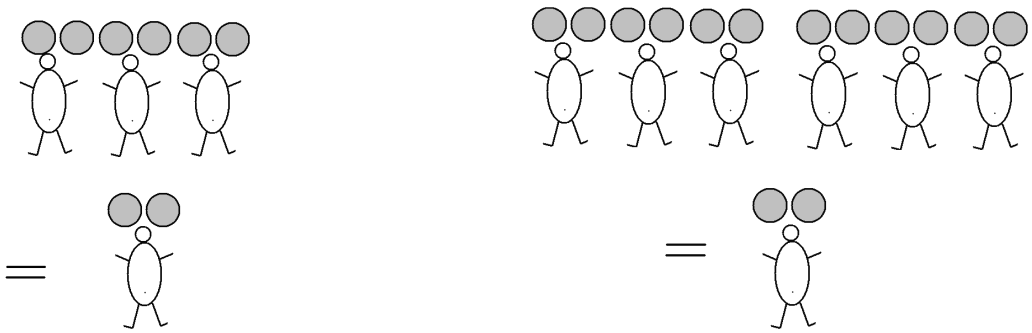
We have that $\frac{a}{b}$ is an answer to a division problem:

> $\frac{a}{b}$ represents the amount of pie an individual boy receives when a pies are distributed among b boys.

What happens if we double the number of pies and double the number of boys? Nothing! The amount of pie per boy is still the same:

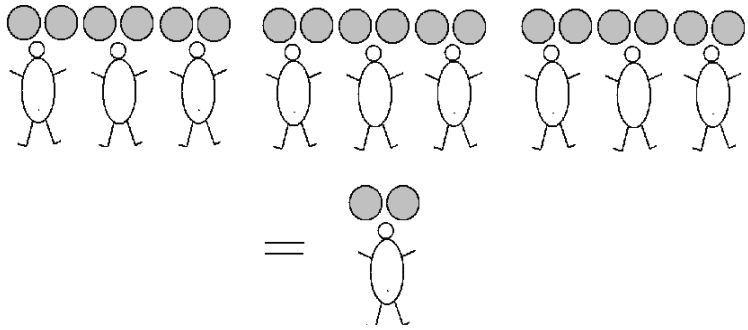
$$\frac{2a}{2b} = \frac{a}{b}.$$

For example, as the picture shows, $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies for each boy.



And tripling the number of pies and tripling the number of boys also does not change the final amount of pie per boy, nor does quadrupling each number, or one-trillion-billion-tupling the numbers!

$$\frac{6}{3} = \frac{12}{6} = \frac{18}{9} = \cdots = \text{two pies per boy}$$



This leads us to want to believe a fraction rule:

FRACTION RULE: $\frac{xa}{xb} = \frac{a}{b}$ (for positive whole numbers at least).

For example,

$$\frac{3}{5} \quad \text{(sharing three pies among five boys)}$$

yields the same result as

$$\frac{3 \times 2}{5 \times 2} = \frac{6}{10} \quad \text{(sharing six pies among ten boys),}$$

and as

$$\frac{3 \times 100}{5 \times 100} = \frac{300}{500} \quad \text{(sharing } 300 \text{ pies among } 500 \text{ boys).}$$

Going backwards …

$$\frac{20}{32} \quad \text{(sharing } 20 \text{ pies among } 32 \text{ boys)}$$

is the same problem as:

$$\frac{5 \times 4}{8 \times 4} = \frac{5}{8} \quad \text{(sharing five pies among eight boys).}$$

Comment: Most people say we have "cancelled" or "taken" a common factor of 4 from the numerator and the denominator.

Mathematicians call this process *reducing* the fraction to simpler terms. (We've made the numerator and denominator each smaller.) Teachers tend to say that we are *simplifying* the fraction. (One has to admit that $\frac{5}{8}$ does look simpler than $\frac{20}{32}$.)

As another example $\frac{280}{350}$ can certainly be simplified by noticing that there is a common factor of 10 in both the numerator and the denominator:

$$\frac{280}{350}=\frac{28\times 10}{35\times 10}=\frac{28}{35}.$$

We can go further as 28 and 35 are both multiples of 7:

$$\frac{28}{35}=\frac{4\times 7}{5\times 7}=\frac{4}{5}.$$

Thus, sharing 280 pies among 350 boys gives the same result as sharing just 4 pies among 5 boys!

$$\frac{280}{350}=\frac{4}{5}.$$

As 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers!).

EXERCISE 17: MIX AND MATCH

On the top are some fractions that have not been simplified. On the bottom are the simplified answers, but in random order. Which simplified answer goes with which fraction? (Notice that there are less answers than questions!)

1. $\frac{10}{20}$ **2.** $\frac{50}{75}$ **3.** $\frac{24000}{36000}$ **4.** $\frac{24}{14}$ **5.** $\frac{18}{32}$ **6.** $\frac{1}{40}$

A. $\frac{2}{3}$ **B.** $\frac{9}{16}$ **C.** $\frac{12}{7}$ **D.** $\frac{1}{40}$ **E.** $\frac{1}{2}$

1. goes with _____
2. goes with _____
3. goes with _____
4. goes with _____
5. goes with _____
6. goes with _____

EXERCISE 18: Jenny says that $\frac{4}{5}$ does "reduce" further is you are willing to move away from whole numbers. She writes:

$$\frac{4}{5} = \frac{2\times 2}{2\frac{1}{2}\times 2} = \frac{2}{2\frac{1}{2}}.$$

Is she right? Does sharing 4 pies among 5 boys yield the same result as sharing 2 pies among $2\frac{1}{2}$ boys? What do you think?

ADDING FRACTION IS CONFUSING IN MODEL 2

Here are two very similar fractions: $\frac{2}{7}$ and $\frac{3}{7}$. What might it mean to add them?

Now …

$\frac{2}{7}$ represents 2 pies being shared among 7 boys

$\frac{3}{7}$ represents 3 pies being shared among 7 boys

so $\frac{2}{7}+\frac{3}{7}$ probably represents sharing 5 pies among 14 boys, giving the answer $\frac{5}{14}$.

But this contradicts what model 1 says: two sevenths plus three sevenths equals five sevenths.

Hmm.

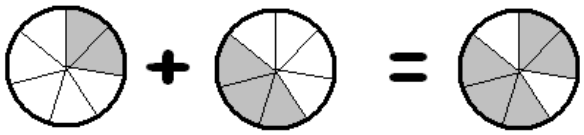
Question: If you are interested … Is there a way to make sense of $\frac{2}{7}+\frac{3}{7}$ equaling $\frac{5}{7}$ in this pie per boy model? (If Poindexter is part of an action of sharing two pies among seven boys and then later part of an action of sharing three pies among seven boys, then …?)

Is this type of think too confusing and not worth the bother?

MULTIPLYING FRACTIONS MAKES NO SENSE IN MODELS 1 AND 2!

At some point in our schooling we learn to multiply fractions. But this is mighty odd.

In model 1 it makes sense to add pie:



but it makes no sense to multiple pie.

In model 2, adding amounts of pie per boy is hard to think through. I have no clue what it means to multiply amount of pie per boy!

BUT ONE TYPE OF MULTIPLICATON DOES MAKE SENSE IN MODEL 2

In the fraction $\frac{a}{b}$, a pies are being shared among b boys.

How could I double the amount of pie each boy receives?
<u>Answer</u>: Double the number of pies!

We have:

$$\frac{2a}{b} = 2 \times \frac{a}{b}$$

Comment: This is really saying something! It reads "If we double the number of pies, then we get double the original amount of pie per boy."

In the same way we can triple the amount of pie per boy by tripling the number of pies: $3 \times \frac{a}{b} = \frac{3a}{b}$, and so on.

This leads to the belief:

$$x \times \frac{a}{b} = \frac{xa}{b}$$

for positive whole numbers x, a, and b.

MODEL 3:
FRACTIONS ON THE NUMBER LINE

In early school we often associate multiplication with the geometry of area.

What is 17×18 *?*

It is the area of a rectangle with one side of length 17 *and the other side of length* 18*.*

(After all, we feel that 306 unit squares will fit into this rectangle.)

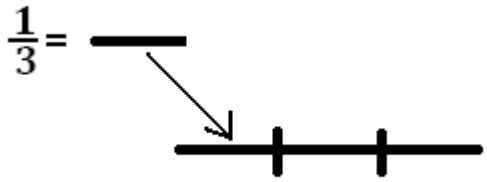
So to multiply fractions, this means we are going to have to – somehow – think of fractions as lengths.

To do this, curriculum writers suggest that we go back to model 1. But instead of thinking of parts of pie, think of parts of the number line.

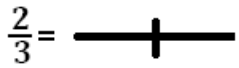
Let's take this slowly:

A pie is now a line segment one unit long.

The fraction $\frac{1}{3}$, say, is a piece of this pie given by dividing that pie into three equal parts.

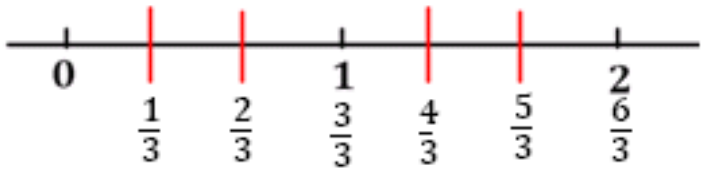


The fraction $\frac{2}{3}$ is shorthand for two of those pieces: $\frac{2}{3} = 2 \times \frac{1}{3}$.



And $\frac{5}{3}$ is five of those pieces, and so on.

Now it seems very tempting to use the "one third" piece as a unit of measure along a number line, measuring to the right from zero.



So now we associate with each fraction a location on the number line. Moreover, each location also represents a length: The distance of that location from zero.

In this model:

A fraction $\frac{a}{b}$ is a location on the number line. It is given by taking a steps from zero in units of $\frac{1}{b}$.

EXERCISE 19: Draw a similar picture to show the location of the point $\frac{7}{5}$ on the number line.

EXERCISE 20: What do you think we mean by the fraction $-\frac{7}{5}$? Can you draw a number-line picture for it?

And moreover we can now associate with each fraction a length:

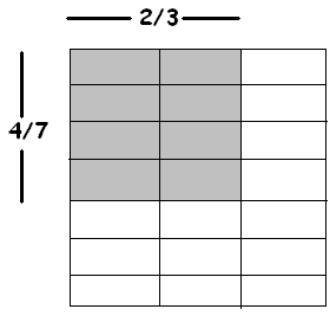
The fraction $\frac{a}{b}$ is a length: It is the distance between 0 and $\frac{a}{b}$ on the number line.

Now that we can associate lengths with fractions, we can now use area to make sense of multiplying fractions. (This seems awfully contorted!)

MULTIPLYING FRACTIONS
THE HARD CONFUSING WAY WITH MODEL 3

Let's work out $\frac{4}{7}\times\frac{2}{3}$, say, as an area problem.

Instead of starting with a rectangle, let's start with square, viewing its sides as unit lengths on a number line. Divide one side-length into sevenths and the other side-length into thirds and mark off the $\frac{4}{7}$ and $\frac{2}{3}$ positions, as though the sides really are part of a number line.



The product $\frac{4}{7}\times\frac{2}{3}$ is the area of the shaded region shown. But we see that the whole square is divided into 21 pieces in all and we've shaded 8 of them. This is $\frac{8}{21}$ of pie.

I'M CONFUSED! I thought we **weren't** using the pie model any more!

THINKING EXERCISE: The area problem $\frac{4}{7}\times\frac{2}{3}$ yielded a diagram with 21 small rectangles, eight of which were shaded. Is it a coincidence that "21" happens to equal seven times three and "8" is four times two?

THE TRUTH ABOUT FRACTIONS: FIVE BELIEFS

In these notes we've looked at three common models for introducing fractions – one has fractions as actual parts of a whole (pieces of pie), one has fractions as proportions (portions of pie per boy), one has fractions as points on the number line (representing lengths: distances from the point zero). There are other models for fractions too. But no one model can speak the whole truth about fractions: the addition of fractions might make sense in one model, but not in another. The same for the multiplication of fractions, or the division, and so on.

But one thing we haven't explicitly said in these notes so far is that:

Fractions should be numbers!

That we represented fractions as locations on the number line came close to making that claim, but we didn't actually make it.

So if we come to idea that fractions should be numbers, we have to ask … *What properties of arithmetic do we feel they should follow?*

We certainly expect them to follow all the usual rules of arithmetic (that a fraction times 1, like all numbers, should remain the same; that the order in which one chooses to multiply two fractions should not matter; and so on). But what special properties to fractions do we feel should be true?

The three models did identify five particular beliefs that feel very right for fractions:

Two basic beliefs:

BELIEF 1: $\frac{a}{a} = 1$

BELIEF 2: $\frac{a}{1} = a$

A key belief:

BELIEF 3: $\frac{ax}{bx} = \frac{a}{b}$

(This one seems important as it explains why $\frac{4}{6}$ and $\frac{2}{3}$, for example, represent the same fraction.)

A belief about basic addition:

BELIEF 4: $\frac{a}{N}+\frac{b}{N}=\frac{a+b}{N}$

And a belief related to multiplication:

BELIEF 5: $x\times\frac{a}{b}=\frac{xa}{b}$

These beliefs, in our models, assume that a, b, x, and N are positive whole numbers.

Even though I don't know what a fraction actually is – none of the models have really pinned down an answer to that question – I have at least identified features that my intuition says should be true about how these numbers, called fractions, whatever they are , should behave.

Now… let's push matters further.

The five beliefs identified here were motivated by models using positive whole numbers. So here's a question:

> *Do these five beliefs feel so fundamental and so right that you feel they should hold for all types of numbers, not necessarily just positive whole numbers?*

It seems delightfully compelling to answer YES just to see where the logical consequences take us!

In fact, if we choose to answer YES to this question we will see that <u>everything</u> that was taught to us about the arithmetic of fractions in school actually follows BY PURE LOGICAL REASONING as CONSEQUENCES of these five beliefs. There is no need to try to come up with different models to try to explain different aspects of work with fractions. Everything follows from these five beliefs.

So here is the best answer I have to the question: WHAT IS A FRACTION?

> **Fractions are some kind of numbers that satisfy the basic five beliefs stated above (and satisfying all the usual rules of arithmetic too). Everything you want to do and understand about fractions follows from these five rules.**

Let's see now how everything does indeed unfold from these five belies.

SOME FIRST CONSEQUENCES:

Some might feel that we should add to our list of beliefs:

"BELIEF 6": $b \times \frac{a}{b} = a$

For example, it says $3 \times \frac{5}{3} = 5$ and $7 \times \frac{1}{7} = 1$.

It need not be a new belief. It follows from the first five as follows:

$$\begin{aligned} b \times \frac{a}{b} &= \frac{b \times a}{b} \quad \text{(by Belief 5)} \\ &= \frac{b \times a}{b \times 1} \\ &= \frac{a}{1} \text{ (by Belief 3)} \\ &= a \text{ (by Belief 2)} \end{aligned}$$

So far all our numerators have been positive whole numbers. What if the numerator is zero?

What is the value of $\frac{0}{5}$?

In the pies per boy thinking of model 2 this is zero pies for five boys. This model suggests the answer is zero. BUT WE DON'T NEED TO RELY ON THE MODELS!

Here's a way, by logic, to see the value of $\frac{0}{5}$:

$$\begin{aligned} \frac{0}{5} &= \frac{0 \times 1}{5} \\ &= 0 \times \frac{1}{5} (\textit{by Belief } 5) \\ &= 0 \end{aligned}$$

Here we used, twice, the basic rule of arithmetic that any number multiplied by zero should be zero.

DIVIDING BY ZERO? A DENOMINATOR OF ZERO?

"Belief 6" says that $b \times \frac{a}{b} = a$. This provides a check to see if a fraction calculation is correct.

For example, suppose I am having trouble computing $\frac{20}{4}$. I think this is 3.

Now "Belief 6" says: $4 \times \frac{20}{4} = 20$. My guess of 3 is not right because $4 \times 3 \neq 20$.

In fact, in general, one checks whether or not a division problem is correct by performing multiplication. For example:

$\frac{6}{2} = 3$ is correct because 2 times 3 is indeed 6.

$\frac{20}{4} = 5$ is correct because 4 times 5 is indeed 20.

$\frac{83}{9} = 11$ is not correct because 9 times 11 is not 83.

$\frac{18}{0.1} = 180$ is correct because 0.1 times 180 is indeed 18.

These are all just uses of "Belief 6." (And remember … We're no longer saying we have to stick with positive whole numbers!)

A THINKING QUESTION:

a) Cyril says that $\frac{5}{0}$ equals 2. Why is he not correct?

b) Ethel says that $\frac{5}{0}$ equals 17. Why is she not correct?

c) Wonhi says that $\frac{5}{0}$ equals 887231243. Why is he not correct?

d) Duane says that there is no answer to $\frac{5}{0}$. Explain why he is correct.

A SECOND THINKING QUESTION:

Cyril says that $\frac{0}{0}$ equals 2.

Ethel says that $\frac{0}{0}$ equals 17.

Wonhi says that $\frac{0}{0}$ equals 887231243.

Why do they each believe that they are correct?

What might Duane say here?

To answer these questions …

Notice that if $\frac{5}{0}=2$, as Cyril says, then we should have that 2 times 0 is 5, according to the check. This is not correct. In fact, the check shows that there is no number x for which $\frac{5}{0}=x$.

On the other hand, Cyril says that $\frac{0}{0}=2$ and he believes he is correct because it passes the check: 2 times 0 is indeed zero. But so too do $\frac{0}{0}=17$ and $\frac{0}{0}=887231243$ pass the check! In fact, $\frac{0}{0}=x$ passes the check for <u>any</u> number x.

The trouble with $\frac{a}{0}$ (with a not zero) is that there are no meaningful values to assign to it, and the trouble with $\frac{0}{0}$ is that there are too many possible values to give it!

In general, most people would say that dividing by zero is "undefined." There is no means to give either an answer that is consistent with the arithmetic. **"Belief 6" suggests we can't allow the denominator of a fraction to be zero.**

FRACTIONS WITH NEGATIVE NUMERATORS AND DENOMINATORS

Mathematically, "-2" represents the opposite of "2", in the sense that adding 2 and -2 together gives zero.

The usual rules of arithmetic also allow us to think of -2 as $(-1)\times 2$ if we prefer.

Here's a confusing question:

Are $\frac{-3}{5}$ and $-\frac{3}{5}$ and $\frac{3}{-5}$ the same fraction or are they all different as numbers?

It is hard to answer this question in any of our models. (Is $\frac{-3}{5}$ the sharing of three anti-pies to five boys, and $\frac{3}{-5}$ the sharing of three pies to five anti boys? Huh?)

But look, using $-a=(-1)\times a$ twice we have:

$$\begin{aligned}\frac{-3}{5} &= \frac{(-1)\times 3}{5}\\ &= (-1)\times\frac{3}{5} \quad (by\ Belief\ 5)\\ &= -\frac{3}{5}.\end{aligned}$$

Also,

$$\frac{-3}{5}=\frac{-3\times(-1)}{5\times(-1)}=\frac{3}{-5}$$

by belief 3.

This shows all three quantities are the same number.

People call writing $\frac{-a}{b}$ as $-\frac{a}{b}$, and writing $\frac{a}{-b}$ as $-\frac{a}{b}$, as "pulling out a negative sign."

EXERCISE 21:

a) What is $\frac{-a}{-b}$?

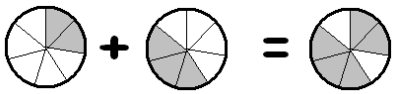
b) What is $\frac{-8}{9} \times \frac{2}{-5}$?

ADDING AND SUBTRACTING FRACTIONS

Belief 4 allows us to add fractions with a common numerator:

$$\frac{a}{N}+\frac{b}{N}=\frac{a+b}{N}$$

It was motivated by the pies of model 1.



$$\frac{2}{7}+\frac{3}{7}=\frac{5}{7}.$$

We can apply this technique to a sum of more than two fractions. For example:

$$\begin{aligned}\frac{4}{10}+\frac{3}{10}+\frac{8}{10}&=\left(\frac{4}{10}+\frac{3}{10}\right)+\frac{8}{10}\\&=\frac{7}{10}+\frac{8}{10}\\&=\frac{15}{10}\\&=\frac{3\times 5}{2\times 5}=\frac{3}{2}\end{aligned},$$

And we can do subtraction too as this is just the addition of the opposite!

$$\begin{aligned}\frac{6}{11}-\frac{2}{11}&=\frac{6}{11}+\left(-\frac{2}{11}\right)\\&=\frac{6}{11}+\frac{-2}{11}\\&=\frac{4}{11}\end{aligned}$$

All is looking grand.

Except … How do we add fractions with no common denominator?

What is $\frac{2}{5}+\frac{1}{3}$, for example?

Belief 3 comes to the rescue!

Write $\frac{2}{5}$ in a series of alternative forms using belief 3, and do the same for $\frac{1}{3}$:

$$\frac{2}{5} \quad + \quad \frac{1}{3}$$

$$\frac{4}{10} \qquad \frac{2}{6}$$

$$\frac{6}{15} \qquad \frac{3}{9}$$

$$\frac{8}{20} \qquad \frac{4}{12}$$

$$\frac{10}{25} \qquad \frac{5}{15}$$

$$\vdots \qquad \vdots$$

We notice two common denominators to see that the problem $\frac{2}{5}+\frac{1}{3}$ is actually the same as $\frac{6}{15}+\frac{5}{15}$, which has answer $\frac{11}{15}$:

$$\frac{2}{5}+\frac{1}{3} = \frac{6}{15}+\frac{5}{15} = \frac{11}{15}.$$

As another example:

$$\frac{3}{8} + \frac{3}{10} = \frac{27}{40}$$

$$\frac{6}{16} \qquad \frac{6}{20}$$

$$\frac{9}{24} \qquad \frac{9}{30}$$

$$\frac{12}{32} \qquad \frac{12}{40}$$

$$\frac{15}{40} \qquad \frac{15}{50}$$

$$\vdots \qquad \vdots$$

COMMENT: Of course one doesn't need to list all the equivalent forms of each fraction in order to find a common denominator. If you can see a common denominator right away (or can think of a method that always works), go for it.

EXERCISE 22: What is $\frac{1}{2}+\frac{1}{3}$? The answer is some number of sixths. How many sixths?

EXERCISE 23: What is $\frac{2}{5}+\frac{37}{10}$?

EXERCISE 24: What is $\frac{1}{2}+\frac{3}{10}$?

EXERCISE 25: What is $\frac{2}{3}+\frac{5}{7}$?

EXERCISE 26: What is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$?

EXERCISE 27: What is $\frac{3}{10}+\frac{4}{25}+\frac{7}{20}+\frac{3}{5}+\frac{49}{50}$?

Let's do subtraction.

EXERCISE 28: What is $\frac{7}{10}-\frac{3}{10}$?

EXERCISE 29: What is $\frac{7}{10}-\frac{3}{20}$?

EXERCISE 30: What is $\frac{1}{3}-\frac{1}{5}$?

EXERCISE 31: What is $\frac{2}{35}-\frac{2}{7}+\frac{2}{5}$?

EXERCISE 32: What is $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}$?

Here's a good question!

EXERCISE 33: Which is larger: $\frac{5}{9}$ or $\frac{6}{11}$?

What is a good way to approach this? Perhaps write each fraction with a common denominator?

MULTIPLYING FRACTIONS

Our five beliefs tell us how to multiply fractions.

Compute $\frac{2}{3}\times\frac{4}{7}$**.**

By Belief 5 this equals $\frac{\frac{2}{3}\cdot 4}{7}$. (Recall: $x\times\frac{a}{b}=\frac{xa}{b}$.)

By Belief 3 we can multiply the numerator and denominator both by 3 without changing the fraction. This gives (using "Belief 6" along the way):

$$\frac{\frac{2}{3}\cdot 4}{7} = \frac{\frac{2}{3}\cdot 4\cdot 3}{7\cdot 3} = \frac{2\cdot 4}{7\cdot 3}.$$

In effect we have simply produced a fraction with numerator the product of the original numerators, and denominator the product of the original denominators:

$$\frac{2}{3}\times\frac{4}{7}=\frac{2\cdot 4}{3\cdot 7}.$$

The general multiplication rule for fractions we learn in early school days is a logical consequence of the beliefs!

MULTIPLICATION RULE: $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$**.**

There is no need to try to "explain" this with number lines drawn on the sides of rectangular pies.

EXERCISE 34: Ibrahim was asked to compute:

$$\frac{18}{7}\times\frac{70}{36}$$

and, within three seconds, said that the answer was 5. He was right! How did he see this so quickly?

EXERCISE 35: What is the value of $\frac{39}{35}\times\frac{14}{13}$?

EXERCISE 36: Compute the following products. (Don't work too hard!)

a) $\frac{3}{4}\times\frac{1}{3}\times\frac{2}{5}$ b) $\frac{5}{5}\times\frac{7}{8}$ c) $\frac{88}{88}\times\frac{541}{788}$ d) $\frac{77876}{311}\times\frac{311}{77876}$

e) $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}\times\frac{6}{7}\times\frac{7}{8}\times\frac{8}{9}\times\frac{9}{10}$

(Make good use of the fraction rule $\frac{xa}{xb}=\frac{a}{b}$ <u>before</u> you do any arithmetic!)

MIXED NUMBERS

People like to name things (for reasons that are not always clear).

A fraction with a numerator smaller than its denominator is called a <u>proper fraction</u>. E.g. $\frac{45}{58}$ is a proper fraction.

A fraction with numerator larger than its denominator is called an <u>improper fraction</u>. E. g. $\frac{7}{3}$ is an improper fraction. (In the 1800s, these fractions were called <u>vulgar fractions</u>. They were considered common.)

For some reason that doesn't really make sense, improper fractions are considered, well, improper by some teachers, and students are made to write improper fractions as a combination of a whole number and a proper fraction.

Consider, for example, $\frac{7}{3}$. If seven pies are shared among three girls, then each girl will certainly receive 2 whole pies, leaving one pie over to share among the three girls. Thus, $\frac{7}{3}$ equals 2 plus $\frac{1}{3}$. People write:

$$\frac{7}{3} = 2\frac{1}{3}$$

and call the result $2\frac{1}{3}$ a <u>mixed number</u>. (One can also write $2+\frac{1}{3}$, which is what $2\frac{1}{3}$ really means, but most people choose to suppress the plus sign.)

QUESTION: We used model 2, pies per girl, to explain why $\frac{7}{3}$ equals $2\frac{1}{3}$. How do we check this using only our five beliefs?

Answer: Does $2\frac{1}{3}$ pass our "belief 6" check?

$$\left(2\frac{1}{3}\right)\times 3=\left(2+\frac{1}{3}\right)\times 3=6+\frac{3}{3}=6+1=7$$

So yes, $2\frac{1}{3}$ is $\frac{7}{3}$.

As another example, consider $\frac{23}{4}$. The number 4 certainly "goes into" 23 five times and leaves a remainder of 3, which is still be divided by four. We have:

$$\frac{23}{4}=5\frac{3}{4}.$$

(And via our Belief 6 check: $4\times 5\frac{3}{4}=4\times\left(5+\frac{3}{4}\right)=20+3=23$, as hoped.)

EXERCISE 37: Write each of the following as a mixed number. (For example, $\frac{32}{5}$ equals $6\frac{2}{5}$.)

a) $\frac{17}{3}$ b) $\frac{8}{5}$ c) $\frac{100}{13}$ d) $\frac{200}{199}$

Mathematically there is nothing wrong with an improper fraction and many mathematicians prefer improper fractions over mixed numbers.

Consider, for instance, the mixed number $2\frac{1}{5}$. This is really $2+\frac{1}{5}$.

For fun, let's write the number 2 as a fraction with denominator five:

$$2=\frac{2}{1}=\frac{2\times 5}{1\times 5}=\frac{10}{5}.$$

(Do you see Beliefs 2 and 3 at play?) So the number $2\frac{1}{5}$ is:

$$2+\frac{1}{5}=\frac{10}{5}+\frac{1}{5}=\frac{11}{5}.$$

We've written the mixed number $2\frac{1}{5}$ as the improper fraction $\frac{11}{5}$.

EXERCISE 38: Convert each of these mixed numbers back into proper fractions:

a) $3\frac{1}{4}$ b) $5\frac{1}{6}$ c) $1\frac{3}{11}$ d) $200\frac{1}{200}$

COMMENT: Students are often asked to memorize the names "proper fraction," "improper fraction" and "mixed number" so that they can follow directions on tests and problem sets. But, to a mathematician, these names are not at all important. There is no "correct" way to express an answer – as long as the answer is mathematically correct!

Just decide for yourself as you do your mathematics which type of fraction would be best to work with for your task at hand.

DIVIDING FRACTIONS

Here is a nasty problem:

> $7\frac{2}{3}$ *pies are to be shared among* $5\frac{3}{4}$ *girls. How many pies per individual girl does this yield?*

Technically, we could just write down the answer as $\dfrac{7\frac{2}{3}}{5\frac{3}{4}}$ and be done! (This is indeed the correct fraction for the problem!) Is there a way to make this look friendlier?

Recall the key fraction rule Belief 3:

$$\boxed{\frac{xa}{xb}=\frac{a}{b}}$$

Let's multiply the numerator and denominator of our answer each by a convenient choice of number. Right now we have the expression:

$$\frac{7\frac{2}{3}}{5\frac{3}{4}}=\frac{7+\frac{2}{3}}{5+\frac{3}{4}}.$$

Let's multiply by 3. (Why three?)

$$\frac{\left(7+\frac{2}{3}\right)\times 3}{\left(5+\frac{3}{4}\right)\times 3}=\frac{21+2}{15+\frac{9}{4}}.$$

(Recall from "Belief 6" that $\frac{a}{b}\times b$ equals a.)

Let's now multiply numerator and denominator each by 4. (Why four?)

$$\frac{(21+2)\times 4}{\left(15+\frac{9}{4}\right)\times 4}=\frac{84+8}{60+9}=\frac{92}{69}$$

We now see that the answer is $\frac{92}{69}$. Sharing $7\frac{2}{3}$ pies among $5\frac{3}{4}$ girls is the same as sharing 92 pies among 69 girls!

As another example, consider $\frac{3\frac{1}{2}}{1\frac{1}{2}}$.

Multiplying the numerator and denominator each by 2 should be enough to make the expression look friendlier:

$$\frac{3\frac{1}{2}}{1\frac{1}{2}}=\frac{3+\frac{1}{2}}{1+\frac{1}{2}}=\frac{\left(3+\frac{1}{2}\right)\cdot 2}{\left(1+\frac{1}{2}\right)\cdot 2}=\frac{6+1}{2+1}=\frac{7}{3}.$$

EXERCISE 39: Make $\frac{4\frac{2}{3}}{5\frac{1}{3}}$ look friendlier.

EXERCISE 40: Make $\frac{2\frac{1}{5}}{2\frac{1}{4}}$ look friendlier.

EXERCISE 41: Make $\frac{1\frac{4}{7}}{2\frac{3}{10}}$ look friendlier.

EXERCISE 42: Make $\dfrac{\frac{3}{7}}{\frac{4}{5}}$ look friendlier.

Without realizing it, we have just learned how to divide fractions.

For example, let's compute $\frac{3}{5} \div \frac{4}{7}$. This is the fraction:

$$\frac{\frac{3}{5}}{\frac{4}{7}}$$

Let's multiply numerator and denominator each by 5:

$$\frac{\frac{3}{5} \times 5}{\frac{4}{7} \times 5} = \frac{3}{\frac{20}{7}}$$

Let's now multiply top and bottom each by 7:

$$\frac{3 \times 7}{\frac{20}{7} \times 7} = \frac{21}{20}$$

Done!

Let's do another. Let's consider $\frac{5}{9} \div \frac{8}{11}$, that is:

$$\frac{\frac{5}{9}}{\frac{8}{11}}$$

Let's multiply top and bottom each by 9 and by 11 at the same time. (Why not?)

$$\frac{\frac{5}{9} \times 9 \times 11}{\frac{8}{11} \times 9 \times 11} = \frac{5 \times 11}{8 \times 9}$$

(Do you see what happened here?)

and so:

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{5 \times 11}{8 \times 9} = \frac{55}{72}.$$

EXERCISE 43: Compute each of the following:

a) $\frac{1}{2} \div \frac{1}{3}$ b) $\frac{4}{5} \div \frac{3}{7}$ c) $\frac{2}{3} \div \frac{1}{5}$

EXERCISE 44: Compute $\frac{45}{45} \div \frac{902}{902}$. Do you see what the answer simply must be?

EXERCISE 45: Compute $\frac{10}{13} \div \frac{2}{13}$. Any general comments about this one?

THINKING EXERCISE:

Consider the problem $\frac{5}{12} \div \frac{7}{11}$.

Janine wrote:

$$\frac{\frac{5}{12}}{\frac{7}{11}} = \frac{\frac{5}{12} \times 12 \times 11}{\frac{7}{11} \times 12 \times 11} = \frac{5 \times 11}{7 \times 12} = \frac{5}{12} \times \frac{11}{7}$$

and then stopped before completing her final step.

a) Check each step of her work here and make sure that she is correct in what she did up to this point.

Janine then exclaimed: "Dividing one fraction by another is the same as multiplying the first fraction with the second fraction upside down."

b) Do you see what Janine means by this from her example?

c) Is she right? Is dividing two fractions always the same as multiplying the two fractions with the second one turned upside down? What do you think?

Work out $\frac{\frac{3}{7}}{\frac{4}{13}}$. Is the answer the same as $\frac{3}{7} \times \frac{13}{4}$?

Work out $\frac{\frac{2}{5}}{\frac{3}{10}}$. Is the answer the same as $\frac{2}{5} \times \frac{10}{3}$?

Work out $\frac{\frac{a}{b}}{\frac{c}{d}}$. Is the answer the same as $\frac{a}{b} \times \frac{d}{c}$?

THINKING EXERCISE:

Some teachers have students solve fraction division by rewriting expressions via a common denominator. For example, to compute:

$$\frac{3}{4} \div \frac{2}{3}$$

rewrite the problem as:

$$\frac{9}{12} \div \frac{8}{12}.$$

The claim is then made that the answer to the original problem is $9 \div 8 = \frac{9}{8}$.

a) Does $\frac{3}{4} \div \frac{2}{3}$ indeed equal $\frac{9}{8}$?

b) Work out $\frac{5}{4} \div \frac{7}{9}$ via the method of this section, and then again by the method described above. Are the answers indeed the same?

Why do you think this "common denominator method" works?

THINKING EXERCISE:

Work out $\frac{12}{15} \div \frac{3}{5}$ and show that it equals $\frac{4}{3}$.

Now notice that

$$12 \div 3 = 4$$
$$15 \div 5 = 3$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3}.$$

Is this a coincidence or does $\frac{a}{b} \div \frac{c}{d}$ always equal $\frac{a \div c}{b \div d}$?

ALGEBRA CONNECTIONS

(for those with upper high school mathematics experience)

In an advanced algebra course students are often asked to work with complicated expressions of the following ilk:

$$\frac{\frac{1}{x}+1}{\frac{3}{x}}.$$

We can make it look friendlier by following exactly the same technique of the previous section. In this example, let's multiply the numerator and denominator each by x. (Do you see why this is a good choice?) We obtain:

$$\frac{\left(\frac{1}{x}+1\right)\times x}{\left(\frac{3}{x}\right)\times x}=\frac{1+x}{3}$$

and $\frac{1+x}{3}$ is much less scary.

As another example, given:

$$\frac{\frac{1}{a}-\frac{1}{b}}{ab}$$

one might find it helpful to multiply the numerator and the denominator each by a and then each by b:

$$\frac{\left(\frac{1}{a}-\frac{1}{b}\right)\times a\times b}{ab\times a\times b}=\frac{b-a}{a^2b^2},$$

and for

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5}$$

it might be good to multiply top and bottom each by $(w+1)^2$:

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5}=\frac{1-2(w+1)^2}{1+5(w+1)^2}$$

EXERCISE 46: Make each of the following expressions look less scary:

a) $\dfrac{2-\frac{1}{x}}{1+\frac{1}{x}}$

b) $\dfrac{\frac{1}{x+h}+5}{\frac{1}{x+h}}$

c) $\dfrac{1}{\frac{1}{a}+\frac{1}{b}}$

d) $\dfrac{\frac{1}{x+a}-\frac{1}{x}}{a}$

e) $\dfrac{1}{s^{-2}}$

MULTIPLYING AND DIVIDING BY NUMBERS BIGGER AND SMALLER THAN ONE

People say that multiplying a quantity by a number bigger than one makes the answer bigger. Is this true?

For instance, $\frac{5}{4}$ represents more than one pie. Does multiplying 100, for example, by $\frac{5}{4}$ give an answer bigger than 100?

Well, yes:

$$\frac{5}{4}\times 100=\frac{500}{4}=125\,.$$

Does multiplying any number, let's call it X, by $\frac{5}{4}$ give an answer larger than X?

The answer is yes, and here it is good to write $\frac{5}{4}$ as a mixed number, $1\frac{1}{4}$, to see why. (Ah … mixed numbers are good for something!)

$$\begin{aligned}\frac{5}{4}\times X &= \left(1+\frac{1}{4}\right)X\\ &= 1\cdot X+\frac{1}{4}\cdot X\\ &= X+\ more.\end{aligned}$$

Yes, the answer is bigger than X.

EXERCISE 47: Show that multiplying a number by $\frac{8}{5}$ is sure to give a larger answer.

EXERCISE 48: Show that multiplying a number by $\frac{20}{9}$ is sure to give a larger answer.

Does multiplying a quantity by a number smaller than one makes the answer smaller?

Consider $\frac{4}{5}$, for instance. This represents less than one pie. Does multiplying 100 by it give a smaller answer?

$$\frac{4}{5}\times 100=\frac{400}{5}=80.$$

Yes!

Does multiplying any number X by $\frac{4}{5}$ give an answer smaller than X?

The answer is yes but we need to be tricky and write $\frac{4}{5}$ as a mixed number in an unusual way.

Notice that $\frac{4}{5}+\frac{1}{5}=1$, and so $\frac{4}{5}=1-\frac{1}{5}$. Thus:

$$\begin{aligned}\frac{4}{5}\times X &= \left(1-\frac{1}{5}\right)X\\ &= X-\frac{1}{5}\cdot X\\ &= \text{ smaller than } X.\end{aligned}$$

EXERCISE 49: Show that multiplying a number by $\frac{7}{8}$ is sure to give a smaller answer.

EXERCISE 50: Show that multiplying a number by $\frac{5}{9}$ is sure to give a smaller answer.

Now let's consider <u>dividing</u> a number by a quantity smaller than one. For example, will 100 divided by $\frac{4}{5}$ give an answer smaller or larger than 100? Let's see:

$$\frac{100}{\frac{4}{5}} = \frac{100 \times 5}{\frac{4}{5} \times 5} = \frac{500}{4} = 125$$

The answer is larger.

In general:

$$\frac{X}{\frac{4}{5}} = \frac{X \times 5}{\frac{4}{5} \times 5} = \frac{5X}{4} = \frac{5}{4} \times X$$

and we know that $\frac{5}{4} \times X$ will be larger than X. (We did this two pages ago.)

EXERCISE 45: Show that dividing a number X by $\frac{7}{9}$ will give an answer larger than X.

EXERCISE 46: Show that dividing a number X by $\frac{8}{5}$ will give an answer <u>smaller</u> than X.

A BRIEF INTRODUCTION TO EGYPTIAN FRACTIONS

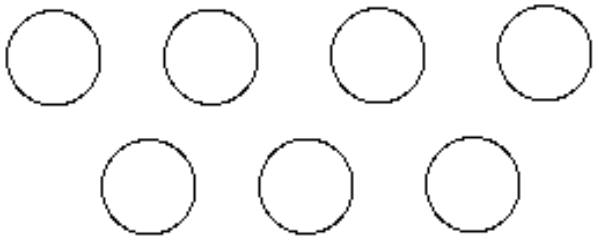
(See *THINKING MATHEMATICS! Vol 1: Arithmetic = The Gateway to All* available at www.lulu.com for more.)

Scholars of ancient Egypt (ca. 3000 B.C.) were very practical in their approaches to mathematics and always sort answers to problems that would be of most convenience to the people involved. This led them to a curious approach to thinking about fractions.

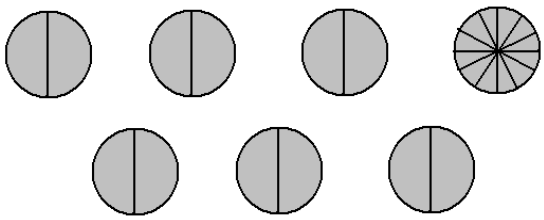
Consider the problem: *Share 7 pies among 12 boys.*

Of course, given our model for fractions, each boy is to receive the quantity "$\frac{7}{12}$" of pie. This answer has little intuitive feel.

But suppose we took this task as a very practical problem. Here are the seven pies:

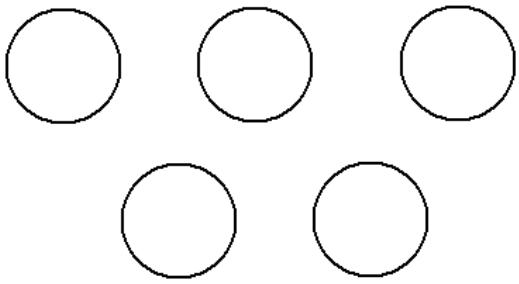


Is it possible to give each of the boys a whole pie? No. How about the next best thing – each boy half a pie? Yes! There are certainly 12 half pies to dole out. There is also one pie left over yet to be shared among the 12 boys. Divide this into twelfths and hand each boy an extra piece.



Thus each boy receives $\frac{1}{2}+\frac{1}{12}$ of a pie and it is indeed true that $\frac{7}{12}=\frac{1}{2}+\frac{1}{12}$.

EXERCISE 53:
a) How do you think the Egyptian's might have shared five pies among six girls?



b) How might they have shared 7 pies among 12 students?

The Egyptians insisted on writing all their fractions as sums of fraction with numerators equal to 1. For example:

$\frac{3}{10}$ was written as $\frac{1}{4}+\frac{1}{20}$

$\frac{5}{7}$ was written as $\frac{1}{2}+\frac{1}{5}+\frac{1}{70}$

That is, to share 3 pies among 10 students, the Egyptians said to give each student one quarter of a pie and one twentieth of a pie.

To share 5 pies among 7 students, the Egyptians suggested giving our half a pie, and one fifth of a pie, and one seventieth of a pie to each student.

EXERCISE 54: It is true that $\frac{4}{13}=\frac{1}{4}+\frac{1}{18}+\frac{1}{468}$. What does this say about how the Egyptians would have shared 4 pies among 13 girls?

Curiously, the Egyptians did not like to repeat fractions. Although it is obviously true that:

$$\frac{2}{5}=\frac{1}{5}+\frac{1}{5}$$

the Egyptians really did think it better to give each person receiving pie piece as large as possible, and so preferred the answer:

$$\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$$

(even though it meant giving out a tiny piece of pie with that bigger piece).

EXERCISE 55: Consider the fraction $\frac{2}{11}$.

a) Show that $\frac{1}{5}$ is bigger than $\frac{2}{11}$.

b) Show that $\frac{1}{6}$ is smaller than $\frac{2}{11}$.

c) Work out $\frac{2}{11}-\frac{1}{6}$.

Use c) to write $\frac{2}{11}$ the Egyptian way.

EXERCISE 56: Consider the fraction $\frac{2}{7}$.

a) What is the biggest fraction $\frac{1}{N}$ that is still smaller than $\frac{2}{7}$?

b) Write $\frac{2}{7}$ the Egyptian way.

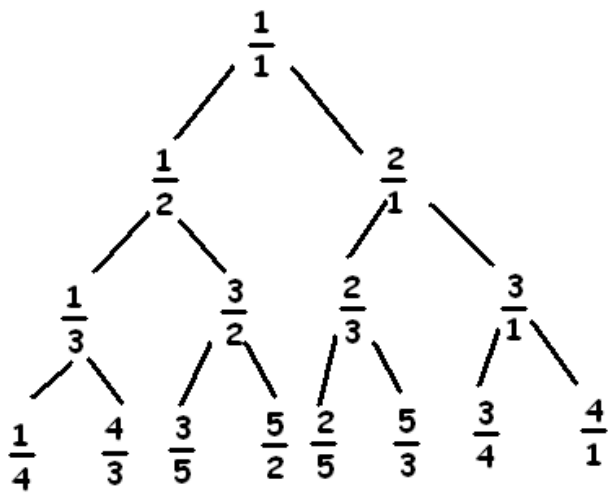
EXERCISE 57: CHALLENGE

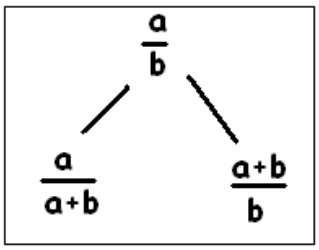
a) Write $\frac{17}{20}$ the Egyptian way.

b) Write $\frac{3}{7}$ the Egyptian way.

A CURIOUS FRACTION TREE

Here is something fun to think about. Consider the following "fraction tree:"





Do you see how it works? Do you see that each fraction has two "children"? The left child is always a number smaller than 1 and the right child is always a number larger than 1.

Do you see how the box to the upper right gives the method for computing the two children of the fraction?

a) Continue the drawing the fraction tree for another two rows.

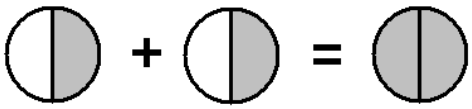
b) Explain why the fraction $\frac{13}{20}$ will eventually appear in the tree. (It might be easier to figure out what $\frac{13}{20}$'s parent is by first noticing that $\frac{13}{20}$ is a "left child." What is its grandparent? What is its great grand parent?)

c) Explain why the fraction $\frac{13}{20}$ cannot appear twice in the tree.

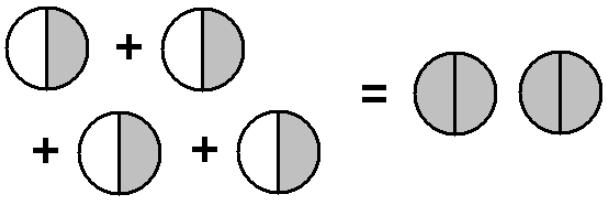
d) Will the fraction $\frac{456}{777}$ eventually appear in the tree? Could it appear twice?

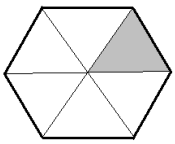
BRIEF SOLUTIONS

1. $\frac{2}{2}$ is two groups of $\frac{1}{2}$. We have:

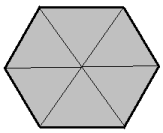


$\frac{4}{2}$ is four groups of $\frac{1}{2}$. We have:

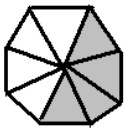


2. a) $\frac{1}{6} =$ 

It is easy to divide a hexagon into six equal parts.

b) $\frac{6}{6} =$ 

c) Octagon

d)  is half the pie.

3. Each "brick" represents $\frac{1}{72}$. Thirty-five are shaded. We have $35 \times \frac{1}{72} = \frac{35}{72}$.

4. This is $5 \times \frac{1}{8} = \frac{5}{8}$.

5. This is correct thinking.

6. $\frac{1}{1}=1$ One pie per boy!

7. $\frac{5}{1}=5$ Five pies for one (lucky) boy!

8. Eight pies for eight pies gives one pie per boy.

9. 3 pies, 2 boys. (Divide each pie into two parts. Each boy get three of those parts.)

10. 8 pies, 3 boys (Divide each pie into three parts. Each boy gets eight of those parts.)

11. $\frac{a}{a}=1$. An equal number of pies as boys gives one pie per boy.

12. $\frac{1876}{1}=1876$

13. $\frac{0}{7}=0$ (Zero pie per boy.)

14. 6

15. Five pies for each half of a boy gives 10 pies for a whole boy.

16. Four pies for each third of a boy gives 12 pies for a whole boy.

17. 1(E) 2(A) 3(A) 4(C) 5(B) 6(D)

18. This is absolutely valid thinking all the way through.

19. Divide first unit of the number line into five equal parts. Use one segment as a unit of measure to measure seven units to the right of zero. This gives a location on the number line between the 1 and 2 mark.

20. This the same as question 19 except it is the portion of the number line extending to the left from zero.

21. a) $\frac{a}{b}$ (Multiply numerator and denominator each by -1) b) $\frac{16}{45}$

22. $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$

23. $\frac{2}{5}+\frac{37}{10}=\frac{4}{10}+\frac{37}{10}=\frac{41}{10}$

24. $\frac{1}{2}+\frac{3}{10}=\frac{5}{10}+\frac{3}{10}=\frac{4}{5}$

25. $\frac{2}{3}+\frac{5}{7}=\frac{14}{21}+\frac{15}{21}=\frac{29}{21}$

26. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{4}{8}+\frac{2}{8}+\frac{1}{8}=\frac{7}{8}$

27. $\frac{30}{100}+\frac{16}{100}+\frac{35}{100}+\frac{60}{100}+\frac{98}{100}=\frac{239}{100}$

28. $\frac{7}{10}-\frac{3}{10}=\frac{2}{5}$

29. $\frac{14}{20}-\frac{3}{20}=\frac{11}{20}$

30. $\frac{5}{15}-\frac{3}{15}=\frac{2}{15}$

31. $\frac{2}{35}-\frac{10}{35}+\frac{14}{35}=\frac{6}{35}$

32. $\frac{8}{16}-\frac{4}{16}-\frac{2}{16}-\frac{1}{16}=\frac{1}{16}$

33. $\frac{5}{9}=\frac{55}{99}$ and $\frac{6}{11}=\frac{54}{99}$ so $\frac{5}{9}$ is larger.

34. $\frac{18\times 70}{7\times 36}=\frac{18\times 7\times 10}{7\times 18\times 2}=\frac{10}{2}=5$

35. $\frac{3\times 2}{5\times 1}=\frac{6}{5}$

36. a) $\frac{3\times 1\times 2}{4\times 3\times 5}=\frac{1\times 2}{4\times 5}=\frac{1\times 1}{2\times 5}=\frac{1}{10}$ b) $\frac{5}{5}\times\frac{7}{8}=1\times\frac{7}{8}=\frac{7}{8}$ c) $\frac{541}{788}$ d) 1 e) $\frac{1}{10}$

37. a) $5\frac{2}{3}$ b) $1\frac{3}{5}$ c) $7\frac{9}{13}$ d) $1\frac{1}{199}$

38. a) $\frac{13}{4}$ b) $\frac{31}{6}$ c) $\frac{14}{11}$ d) $\frac{40001}{200}$

39. $\dfrac{4\frac{1}{3}\times 3}{5\frac{1}{3}\times 3}=\dfrac{12+1}{15+1}=\dfrac{13}{16}$

40. $\dfrac{2\frac{1}{5}\times 5\times 4}{2\frac{1}{4}\times 5\times 4}=\dfrac{40+4}{40+5}=\dfrac{44}{45}$

41. $\dfrac{1\frac{4}{7}\times 7\times 10}{2\frac{3}{10}\times 7\times 10}=\dfrac{70+40}{140+21}=\dfrac{110}{161}$

42. $\dfrac{\frac{3}{7}\times 7\times 5}{\frac{4}{5}\times 7\times 5}=\dfrac{15}{28}$

43. a) $\frac{3}{2}$ b) $\frac{28}{15}$ c) $\frac{10}{3}$

44. $1\div 1=1$

45. $\dfrac{\frac{10}{13}}{\frac{2}{13}}=\dfrac{10}{2}=5$. Is it a coincidence that $\dfrac{a/N}{b/N}=\dfrac{a}{b}$?

46. a) $\dfrac{2x-1}{x+1}$ b) $\dfrac{1+5(x+h)}{1}=1+5(x+h)$ c) $\dfrac{ab}{b+a}$

d) $\dfrac{x-(x-a)}{ax(x-a)}=\dfrac{-a}{ax(x-a)}=-\dfrac{1}{x(x-a)}$ e) $\dfrac{s^2}{s^{-2}s^2}=\dfrac{s^2}{1}=s^2$

47. $\frac{8}{5}\cdot X=\left(1+\frac{3}{5}\right)X=X+\frac{3}{5}X=X+more$

48. $\frac{20}{9}\cdot X=\left(1+\frac{11}{9}\right)X=X+\frac{11}{9}X=X+more$

49. $\frac{7}{8}\cdot X=\left(1-\frac{1}{8}\right)X=X-\frac{1}{8}X=less\ than\ X$

50. $\frac{5}{9}\cdot X=\left(1-\frac{4}{9}\right)X=X-\frac{4}{9}X=less\ than\ X$

51. $\frac{X}{\frac{7}{9}}=\frac{9X}{7}=\frac{9}{7}\cdot X=more\ than\ X$

52. $\frac{X}{\frac{8}{5}}=\frac{5X}{8}=\frac{5}{8}\cdot X=less\ than\ X$

53. a) $\frac{1}{2}+\frac{1}{3}$ Half a pie and a third of a pie to each girl

b) $\frac{1}{2}+\frac{1}{12}$ Half a pie and a twelfth of a pie to each girl.

54. One quarter of a pie and one 18th of a pie and one 468th of a pie to each girl.

55.a) $\frac{1}{5}=\frac{11}{55}$ and $\frac{2}{11}=\frac{10}{55}$ so $\frac{1}{5}$ is larger. b) $\frac{1}{6}=\frac{11}{66}$ and $\frac{2}{11}=\frac{12}{66}$ so $\frac{1}{6}$ is smaller.

c) $\frac{2}{11}-\frac{1}{6}=\frac{12}{66}-\frac{11}{66}=\frac{1}{66}$ d) $\frac{2}{11}=\frac{1}{6}+\frac{1}{66}$

56. a) $\frac{1}{4}$ b) $\frac{2}{7}=\frac{1}{4}+\frac{1}{28}$. (Other answers are possible.)

57. a) $\frac{17}{20}=\frac{1}{2}+\frac{1}{3}+\frac{1}{60}$ (Other answers are possible.) b) $\frac{3}{7}=\frac{1}{3}+\frac{1}{11}+\frac{1}{231}$ (Other answers are possible.)

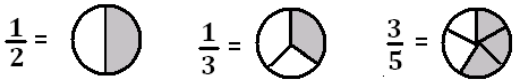
FRACTION TREE: Every reduced fraction does appear in the tree exactly once.

HONESTY STATEMENT:
THE REAL REASON WHY FRACTIONS ARE SO HARD

[This essay appears as the *March 2014 CURRICULUM MATH* essay at www.jamestanton.com/?p=1072 .]

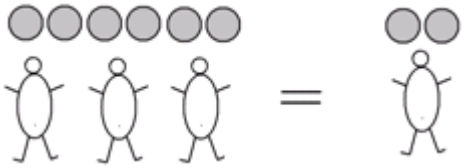
Let me just state at the outset that **fractions are hard!** Mankind has struggled with them for centuries - and rightly so - and we individually struggle with them for years, if not decades, and again, rightly so! It is completely unrealistic - unfair even - to expect students to be comfortable with fractions by the end of grade school, by the end of middle school, or even by the end of high school. If we think about how fractions are introduced and used throughout the standard curriculum, matters are fundamentally confusing and contradictory.

In the early grades, fractions are often introduced as pieces of pie, or parts of some other favorite whole.



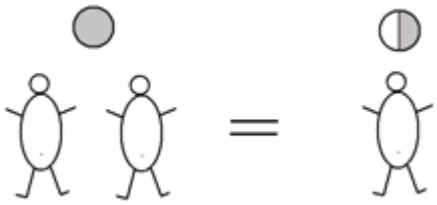
This is often motivated through sharing:

If 6 *pies are shared equally among* 3 *boys, how many pies does each individual boy receive?*



We write: $\frac{6}{3} = 2$ pies per boy.

If 1 *pie is shared equally between* 2 *boys, how much does each individual boy receive?*



We write: $\frac{1}{2} =$ half a pie per boy.

A Point of Confusion: So what is a fraction? Is it an amount of pie or an amount of pie per boy? In my science class we are very fussy about units. What are the units here?

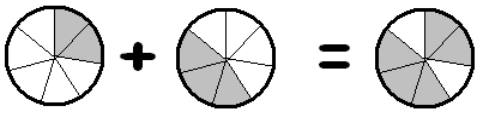
We have right off the bat two slightly different models for what a fraction is.

Model 1: *A fraction is an actual amount of pie one can physically handle.*
Model 2: *A fraction is a proportion of pie per boy.*

Often these models are presented as though they are interchangeable. But there is something unsettling, something hard to articulate, about doing this. Those who think like scientists feel particularly unsettled.

Nonetheless, each of these models is good at motivating certain features we feel ought to be true about fractions.

Model 1 is good for motivating the basic addition of fractions
If I am handling concrete pieces of pie, then drawing pictures of the following ilk feels good and right:



This motivates our belief about how we should add fractions: $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$. Here we are adding sevenths as though we are adding apples.

[Model 2, on the other hand, is very confusing here: What does $\frac{2}{7}+\frac{3}{7}$ mean? If 2 pies are being shared among 7 boys, and 3 pies are being shared among 7 seven boys, is that 5 five pies being shared among 14 boys? Or is it the same seven boys?]

Model 2 is good for motivating several fundamental fraction beliefs
In this model $\frac{a}{b}$ represents the amount of pie an individual boy receives when a pies are distributed among b boys.

Question : What's $\frac{10}{10}$? This is ten pies being shared equally among ten boys. That's one pie per boy. $\frac{10}{10}=1$.

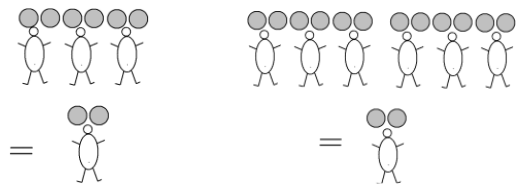
Question: What's $\frac{10}{1}$? That's ten pies being given to one boy. (Lucky boy!) That's ten pies per boy: $\frac{10}{1}=10$.

These suggest, in general, that $\frac{a}{a}=1$ and $\frac{a}{1}=a$.

Suppose we are sharing a pies among b boys. What happens if we double the number of pies and double the number of boys? Nothing! The amount of pie per boy is still the same:

$$\frac{2a}{2b}=\frac{a}{b}.$$

For example, as the picture shows, $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies for each boy.



Tripling the number of pies and tripling the number of boys does not change the final amount of pie per boy, nor does quadrupling each number, or one-trillion-billion-tupling the numbers!

$$\frac{6}{3}=\frac{12}{6}=\frac{18}{9}=\cdots \text{ = two pies per boy}$$

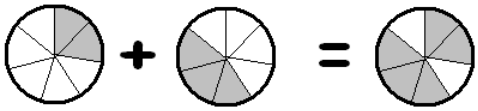
This leads to believe, in general, $\frac{xa}{xb}=\frac{a}{b}$ (at least for positive whole numbers a, b, and x).

A Point of Confusion: Fractions are particularly confusing to young students because they are the first type of number they encounter that are <u>not</u> represented in unique ways. For example, $\frac{15}{20}$ and $\frac{9}{12}$ are the same number even though completely different symbols are being used each time to represent it.

<u>Comment</u>: One can motivate the fraction belief $\frac{xa}{xb}=\frac{a}{b}$ using model 1. (Does that model also motivate $\frac{a}{1}=a$?)

WHERE BOTH MODLES FAIL

In model 1 it makes sense to add pie.



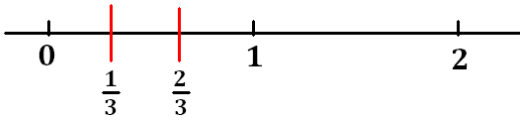
What does it mean to multiply pie?

And I can't even begin to imagine what it means to multiply pies per boy!

After teaching students that fractions are related to pie, it is not uncommon for a curriculum to change track and introduce a third model for fractions, one that let's go of the pie model, but allows for the multiplication of fractions. This is usually done without comment or mention, as though it is "obvious" we are still talking about the same objects.

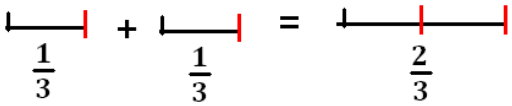
> Model 3: *Fractions are points on the number line (and so are numbers in their own right).*

This is often motivated in a model 2 kind of way: *If I divide the unit interval into three equal parts, what pieces do I see?*



The location labeled $\frac{2}{3}$ is confusing.

We could go back to our model 1 thinking and add pieces of string-like pie:



But is $\frac{2}{3}$ a length of string or a point?

Maybe model 3 should be modified:

> Model 3': *A fraction is a point on the number line, but it actually represents a length – namely the distance between it and the zero point.*

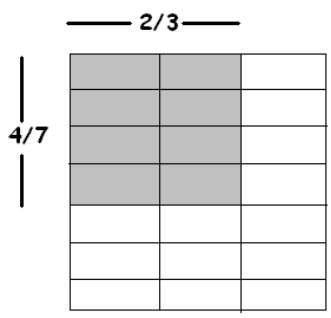
A Point of Confusion: Okay. What about $\frac{7}{3}$? Is that $7 \times \frac{1}{3}$, the unit of $\frac{1}{3}$ added together seven times, or is it the result of dividing a length of seven into three equal parts? I am meant to believe these are the same. Is it obvious that they are?

Assuming we can get past starting hick-ups …

In model $3'$ fractions are lengths and "multiplying lengths" has geometric meaning: we call that a computation of area. So let's use area to motivate the multiplication of fractions. (Poor students who are still thinking pie from the previous year!)

Let's compute $\frac{4}{7} \times \frac{2}{3}$ as an area problem.

Start with a square and divide one side-length into sevenths and the other side-length into thirds and mark off the $\frac{4}{7}$ and $\frac{2}{3}$ positions. (Just as though the sides of the squares are unit lengths on the number line.)



The product $\frac{4}{7} \times \frac{2}{3}$ is the area of the shaded region shown.

But we see that the whole square is divided into 21 pieces in all and we've shaded 8 of them. This is $\frac{8}{21}$ of pie.

HANG ON! I thought we **weren't** using the pie model! (So we do want students to think of the pie model from last year at the same time?)

NO ONE MODEL DOES IT ALL

The reason why fractions are hard is because they are fundamentally an abstract concept. We can model them in different contexts, but no single model will capture all the features we feel hold true for these numbers called fractions.

Multiplication of fractions makes no sense in the concrete pie model 1. But it might make sense if we feel easy about a mix of models 3 and 1.

The addition of fractions is hard to understand in pie-per-boy model 2. (How do you think through $\frac{2}{5}+\frac{4}{3}$?).
But addition does make sense in model 3 – just add lengths of string together.

But model 3 has its limitations in adding fractions: *Okay, I can see what to do geometrically to find the point* $\frac{2}{5}+\frac{4}{3}$*, just lay one length after the other. But what do I do to actually compute its value?*

We present many different models of fractions to students throughout their school years, each with its limitations. And each time we say "This is what a fraction is." But then we abandon the model as soon as we hit a wall (and we always shall hit walls) and switch to a new model and say there: "This is what a fraction is."

Each model is like a blind man feeling an elephant. Each speaks a truth: "An elephant is a flat expanse of leather" (feeling its belly). "An elephant is a hard bone" (feeling the tusk). "An elephant is a length of rope" (feeling its tail). But no model actually says what an elephant is in totality. We've give students different aspects of a truth, but they feel contradictory, hazy at best.

WHAT IS THE TRUTH

As a mathematician I really don't know what a fraction actually is. From a study of abstract algebra I would just say that a fraction is a pair of integers a and b, with b non-zero, usually written above and below a vinculum, $\frac{a}{b}$, with two different expressions $\frac{a}{b}$ and $\frac{c}{d}$ deemed equivalent if $ad = bc$. (Mathematicians formally define what they mean by "equivalent" and what it means to have a whole class of objects deemed equivalent – an "equivalence class.") That is, mathematicians just use what they feel is intuitively true and turn that into the definition and thereby side-step the whole question as to what a fraction actually is!

SO WHAT TO DO FOR STUDENTS?

I am honest with my (middle- and high-school) students. I tell them that I don't know what a fraction actually is, and I invite a discussion about all the things they've seen and have been told about what fractions are. The pie model invariably comes up, as does a sharing model, and the idea of points on the number line. We honestly talk about how helpful/relevant each model is – up to a point – and how each model breaks down, just as I've described in this essay. We talk about whether or not these models are "obviously" equivalent – are they talking about the same things or different things? We wonder if each model is feeling a different part of an elephant. And we conclude in the end that we none of us really knows what a fraction actually is.

But during this discussion we do identify key features of fractions, as suggested by the various models, that seem intuitively right and natural.

For example, we all readily agree that the following two statements certainly feel right:

BELIEF 1: $\frac{a}{a} = 1$
for each positive whole number a.

BELIEF 2: $\frac{a}{1} = a$
for each positive whole number a.

From the basic pie model, adding fractions of like denominator feels natural and right too:

BELIEF 3: $\frac{a}{N} + \frac{b}{N} = \frac{a+b}{N}$
for whole numbers a, b, and N.

From pies per boy, we also have that the following feels natural and right. (It explains why $\frac{8}{10} = \frac{4}{5}$, for example, and so seems important.)

BELIEF 4: $\frac{ax}{bx} = \frac{a}{b}$
for positive whole numbers a, b and x.

At this point, we don't have any fundamental beliefs that mention multiplication. I press one more idea, one from the pies per girl model.

Suppose we share a pies among b girls. (Each girl currently gets $\frac{a}{b}$ pie per girl.) How could we double the amount of pie each girl gets? Answer: Just double the amount of pie!

We have: $\frac{2a}{b} = 2 \times \frac{a}{b}$.

(This really is saying something: doubling the amount of pie doubles the amount of pie per girl.)

As there is nothing special about the number two here, this suggests:

BELIEF 5: $x \times \frac{a}{b} = \frac{xa}{b}$
for positive whole numbers a, b and x.

So the models we experience from grade-school do at least develop an intuitive base for these things we feel exist and are called fractions. And I've pulled out here five basic beliefs that feel particularly fundamental and right.

Continuing to be honest with my students I ask:

Do these beliefs feel so fundamental and so right that you think they should hold for all types of numbers – not just positive whole numbers? Do you want to play the game of exploring the full logical consequences of these beliefs and see where they take us?

I admit we still haven't said what a fraction is – we can't – but we've at least pinned down five pieces of their mathematical behavior.

And the beauty of these five basic properties is that all the remaining properties of fractions that feel familiar to us follow logically from them!

From beliefs 5, 4 and 2 we can prove, for example, that $7 \times \frac{3}{7}$, equals 3. (And in general that $b \times \frac{a}{b} = a$.)

We can use beliefs 4 and 3 to figure out the mathematics of adding fractions: to compute $\frac{3}{7} + \frac{6}{19}$, for example, as a consequence of these beliefs.

We can see how to use beliefs 5 and 4 lead us to multiplying fractions, to compute $\frac{3}{7} \times \frac{6}{19}$ in a snap, just as an mathematical consequence of the beliefs.

We can divide fractions using belief 4 and make the age old rule "to divide, multiply by the reciprocal" obvious and obsolete! (No need to ever say or even think it!)

We can explain why $\frac{-3}{8}$, $-\frac{3}{8}$ and $\frac{3}{-8}$ are all equivalent, using beliefs 4 and 5.

We can explain why $\frac{5}{0}$ has to be undefined and $\frac{0}{0}$ must be too for a different reason.

And so on!

Actually, everything we usually wonder about and want to do mathematically with fractions can be explained and justified as logical consequences of these five basic beliefs!

So, here's my story of fractions for students. During grades K-6 we do, of course, develop an intuitive understanding of fractions and through concrete models, "parts of wholes," and develop some of the mathematics that seems to be appropriate for those models. (Getting to the number of line and thinking in terms of "units of thirds" or "units of fifths," for example, is particularly helpful).

Then, sometime in grade 7 – 12, with a number of models in our minds from the past, we take a moment to reflect on our experiences and come to realize that, actually, it is not clear how all these models "hang" together. We let everything unravel in our reflections, and we explore the deficiencies of the models both individually and as a collective whole. Our job now is to be honest about fractions and admit that actually no one model captures everything we like to believe about them and how they behave.

This then begs the question: *So … what do we believe about fractions and how they should behave?*

We're then upfront about matters and just list our basic beliefs and call them what they are: <u>beliefs</u>!

And with the five particular basic beliefs I've listed pinned down, we marvel at the delight of seeing everything we were taught about the mathematics of fractions just unfold as a series of logical consequences of these basic beliefs.

And if you find a concrete model in which those five beliefs happen to apply - in thinking about lengths on a number line, or in thinking about sharing quantities - then all those mathematical consequences apply to that model too. (And we have to keep in mind that each model will be deficient, and so it will be hard, if not meaningless, to interpret some of the mathematical consequences within that model.)

SO … HOW DOES EVERYTHING FOLLOW FROM THOSE FIVE BASIC BELIEFS?

Rather than let this essay become a tome, let me refer you to:

www.jamestanton.com/?p=1461
(WHICH IS THIS VERY SET OF NOTES!)

for a piece that goes through all those details. Or better yet … Can you figure out for yourself the mathematical logic that explains all the ideas mentioned in the previous page?

I find that middle-school and high-school students enjoy this work. The honest admission that no one, including them, really is meant to "get" fractions from their early introductions to them is an incredible emotional relief. The fractions are abstract is the truth, and students can so handle the truth!