The Theory of Statistical Comparison

with Applications in Quantum Information Science

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Prerequisites

Prerequisites for the first part (general results):

- ✔ basics of probability and information theory: random variables, joint and conditional probabilities, expectation values, etc
- ✔ in particular, noisy channels as probabilistic maps between two sets $w : \mathscr{A} \to \mathscr{B}$: given input $a \in \mathscr{A}$, the probability to have output $b \in \mathscr{B}$ is given by conditional probability $w(b|a)$
- ✔ basics of quantum information theory: Hilbert spaces, density operators, ensembles, POVMs, quantum channels $\equiv$ CPTP maps, composite systems and tensor products, etc

Prerequisites for the second part (applications):

- ✔ resource theories, in particular, quantum thermodynamics: idea of the general setting and of the problem treated (in particular, some knowledge of majorization theory is helpful)
- ✔ entanglement and quantum nonlocality: general ideas such as Bell inequalities, nonocal games, entangled states, etc
- ✔ open systems dynamics: basic ideas such as reduced dynamics, Markov chains and Markovian evolutions, divisibility, etc (quantum case only sketched, see references)

Part I

Statistical Comparison: General Results

Statistical Games (*aka* Decision Problems)

- ✔ **Definition.** A statistical game is a triple $(\Theta, \mathscr{U}, \ell)$, where $\Theta = \{\theta\}$ and $\mathscr{U} = \{u\}$ are finite sets, and ℓ is a function $\ell(\theta, u) \in \mathbb{R}$.
- ✔ **Interpretation.** We assume that θ is the value of a parameter influencing what we observe, but that cannot be observed "directly." Now imagine that we have to choose an action u, and that this choice will earn or cost us $\ell(\theta, u)$. For example, θ is a possible medical condition, u is the choice of treatment, and $\ell(\theta, u)$ is the overall "efficacy."
- ✔ **Resource.** Before choosing our action, we are allowed "to spy" on θ by performing an experiment (i.e., visiting the patient). Mathematically, an experiment is given as a sample set $\mathscr{X} = \{x\}$ (i.e., observable symptoms) together with a conditional probability $w(x|\theta)$ or, equivalently, a family of distributions $\{w_\theta(x)\}_{\theta\in\Theta}$.
- ✔ **Probabilistic decision.** The choice of an action can be probabilistic (i.e., patients with the same symptoms are randomly given different therapies). Hence, a decision is mathematically given as a conditional probability $d(u|x)$.

$$
\begin{array}{ccccccc}
\Theta & \xrightarrow{\text{experiment}} & \mathscr{X} & \xrightarrow{\text{decision}} & \mathscr{U} & & \\
\wr & & \wr & & \wr & \Longrightarrow & \ell(\theta, u) \\
\theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u & &
\end{array}
$$

- ✔ **Example in information theory.** Imagine that θ is the input to a noisy channel, x is the output we receive, and u is the message we decode.

How much is an experiment worth?

$$\begin{array}{ccccccc} \Theta & \xrightarrow{\text{experiment}} & \mathscr{X} & \xrightarrow{\text{decision}} & \mathscr{U} & & \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & \Longrightarrow & \ell(\theta, u) \\ \theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u & & \end{array}$$

- ✔ experiments help us choosing the action "sensibly." How much would you pay for an experiment?
- ✔ **Expected payoff.** $\mathbb{E}_\ell[w] \triangleq \max_{d(u|x)} \sum_{u,x,\theta} \ell(\theta, u) d(u|x) w(x|\theta) \frac{1}{|\Theta|}$. (Bayesian assumption for simplicity, but this is not necessary.)
- ✔ consider now a different experiment (but about the same unknown parameter θ) with sample set $\mathscr{Y} = \{y\}$ and conditional probability $w'(y|\theta)$. Which is better between $w(x|\theta)$ and $w'(y|\theta)$?
- ✔ such questions are considered in the theory of statistical comparison: a very deep field of mathematical statistics, pioneered by Blackwell and greatly developed by Le Cam and Torgersen, among others.
- ✔ **Today's tutorial.** Basic results of statistical comparison, some quantum generalizations, and finally some applications (quantum thermodynamics, quantum nonlocality, open quantum systems dynamics).

Comparison of Experiments: Blackwell's Theorem (1953)

- ✔ **Assumption.** We compare experiments about the same unknown parameter θ

Definition (Information Ordering)

We say that $w(x|\theta)$ is more informative than $w'(y|\theta)$, in formula, $w(x|\theta) \succ w'(y|\theta)$, if and only if $\mathbb{E}_\ell[w] \geqslant \mathbb{E}_\ell[w']$ for all statistical games $(\Theta, \mathscr{U}, \ell)$.

- ✔ **Remark 1.** In the above definition, Θ is fixed, while $\mathscr{U}$ and ℓ vary: the relation $\mathbb{E}_\ell[w] \geqslant \mathbb{E}_\ell[w']$ must hold for all choices of $\mathscr{U}$ and ℓ.
- ✔ **Remark 2.** The ordering $\succ$ is partial.

Theorem (Blackwell, 1953)

$w(x|\theta) \succ w'(y|\theta)$ *if and only if there exists a conditional probability* $\varphi(y|x)$ *such that*

$$w'(y|\theta) = \sum_x \varphi(y|x) w(x|\theta) \; .$$

- ✔ as a diagram:

$$\begin{tikzcd}
\Theta \arrow[r, "\text{experiment}"] \arrow[d, squiggly] & \mathscr{X} \arrow[r, "\text{noise}"] \arrow[d, squiggly] & \mathscr{Y} \arrow[r, "\text{decision}"] \arrow[d, squiggly] & \mathscr{U} \arrow[d, squiggly] & \\
\theta \arrow[r, "w(x|\theta)"'] & x \arrow[r, "\varphi(y|x)"'] & y \arrow[r, "d(u|y)"'] & u & \Longrightarrow \ell(\theta, u)
\end{tikzcd}$$

<table>
<tr><th>classical case</th><th>quantum case</th></tr>
<tr><td>

- statistical game $(\Theta, \mathscr{U}, \ell)$
- sample set $\mathscr{X}$
- experiment $w = \{w_\theta(x)\}$
- probabilistic decision $d(u|x)$
- $p_c(u,\theta) = \sum_x d(u|x) w(x|\theta) \frac{1}{|\Theta|}$
- $\mathbb{E}_\ell[w] = \max_{d(u|x)} \sum \ell(\theta,u) p_c(u,\theta)$

</td><td>

- statistical game $(\Theta, \mathscr{U}, \ell)$
- Hilbert space $\mathcal{H}_S$
- ensemble $\mathcal{E} = \{\rho_S^\theta\}$
- POVM (measurement) $\{P_S^u\}$
- $p_q(u,\theta) = \mathrm{Tr}\left[\rho_S^\theta \ P_S^u\right] \frac{1}{|\Theta|}$
- $\mathbb{E}_\ell[\mathcal{E}] = \max_{\{P_S^u\}} \sum \ell(\theta,u) p_q(u,\theta)$

</td></tr>
<tr><td>

$$\begin{tikzcd} \Theta \arrow[r, "\text{experiment}"] \arrow[d, rightsquigarrow] & \mathscr{X} \arrow[r, "\text{decision}"] \arrow[d, rightsquigarrow] & \mathscr{U} \arrow[d, rightsquigarrow] \\ \theta \arrow[r] & x \arrow[r] & u \end{tikzcd}$$

</td><td>

$$\begin{tikzcd} \Theta \arrow[r, "\text{ensemble}"] \arrow[d, rightsquigarrow] & \mathcal{H}_S \arrow[r, "\text{POVM}"] \arrow[d, rightsquigarrow] & \mathscr{U} \arrow[d, rightsquigarrow] \\ \theta \arrow[r] & \rho_S^\theta \arrow[r] & u \end{tikzcd}$$

</td></tr>
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✔ **Remark.** The same statistical game $(\Theta, \mathscr{U}, \ell)$ can be played with classical resources (statistical experiments and decisions) or quantum resources (ensembles and POVMs).

Comparison of Quantum Ensembles (Vanilla Version)

- ✔ consider now another ensemble $\mathcal{E}' = \{\sigma_{S'}^{\theta}\}$ (different Hilbert space $\mathcal{H}_{S'}$, different density operators, but same parameter set Θ)

Definition (Information Ordering)

We say that $\mathcal{E} = \{\rho_S^{\theta}\}$ is more informative than $\mathcal{E}' = \{\sigma_{S'}^{\theta}\}$, in formula, $\mathcal{E} \succ \mathcal{E}'$, if and only if $\mathbb{E}_\ell[\mathcal{E}] \geqslant \mathbb{E}_\ell[\mathcal{E}']$ for all statistical games $(\Theta, \mathscr{U}, \ell)$.

- ✔ given ensemble $\mathcal{E} = \{\rho_S^{\theta}\}$, define the linear subspace $\mathcal{E}_{\mathbb{C}} \triangleq \{\sum_\theta c_\theta \rho_S^\theta : c_\theta \in \mathbb{C}\} \subseteq \mathsf{L}(\mathcal{H}_S)$

Theorem (Vanilla Quantum Blackwell's Theorem)

$\mathcal{E} \succ \mathcal{E}'$ if and only if there exists a linear, hermitian-preserving, trace-preserving map $\mathcal{L} : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that:

1. *for all $\theta \in \Theta$, $\mathcal{L}(\rho_S^\theta) = \sigma_{S'}^\theta$*
2. *$\mathcal{L}$ is positive on $\mathcal{E}_{\mathbb{C}}$: if $P_S \in \mathcal{E}_{\mathbb{C}}$ is positive semidefinite, i.e., $P_S \geqslant 0$, then $\mathcal{L}(P_S) \geqslant 0$*

- ✔ **Side remark.** In fact, the map $\mathcal{L}$ is somewhat more than just positive on $\mathcal{E}_{\mathbb{C}}$: it is a quantum statistical morphism on $\mathcal{E}_{\mathbb{C}}$. In general:

$$\text{PTP on } \mathsf{L}(\mathcal{H}_S) \underset{\not\Leftarrow}{\Longrightarrow} \text{stat. morph. on } \mathcal{E}_{\mathbb{C}} \underset{\not\Leftarrow}{\Longrightarrow} \text{PTP on } \mathcal{E}_{\mathbb{C}}$$

Quantum Ensembles versus Classical Experiments (Semiclassical Version)

- ✔ **Reminder.** Any statistical game $(\Theta, \mathscr{U}, \ell)$ can be played with classical resources (statistical experiments and decisions) or quantum resources (ensembles and POVMs)
- ✔ we can hence compare a quantum ensemble $\mathcal{E} = \{\rho_S^\theta\}$ with a classical statistical experiment $w = \{w_\theta(x)\}$

Theorem (Semiquantum Blackwell's Theorem)

$\{\rho_S^\theta\} \succ \{w_\theta(x)\}$ if and only if there exists a POVM $\{P_S^x\}$ such that $w_\theta(x) = \mathrm{Tr}\left[P_S^x \ \rho_S^\theta\right]$, for all $\theta \in \Theta$ and all $x \in \mathscr{X}$.

Equivalent reformulation

Consider two ensembles $\mathcal{E} = \{\rho_S^\theta\}$ and $\mathcal{E}' = \{\sigma_{S'}^\theta\}$ and assume that the σ's all commute. Then, $\mathcal{E} \succ \mathcal{E}'$ if and only if there exists a quantum channel (CPTP map) $\Phi : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that $\Phi(\rho_S^\theta) = \sigma_{S'}^\theta$, for all $\theta \in \Theta$.

- ✔ as a diagram:

$$\begin{tikzcd}
\Theta \arrow[r, "\text{ensemble}"] \arrow[d, rightsquigarrow] & \mathcal{H}_S \arrow[r, "\text{quantum noise}"] \arrow[d, rightsquigarrow] & \mathcal{H}_{S'} \arrow[r, "\text{POVM}"] \arrow[d, rightsquigarrow] & \mathscr{U} \arrow[d, rightsquigarrow] \\
\theta \arrow[r, "\mathcal{E}"'] & \rho_S^\theta \arrow[r, "\Phi"'] & \sigma_{S'}^\theta \arrow[r, "\{Q_{S'}^u\}"'] & u
\end{tikzcd}$$

- consider two parameter sets, $\Theta=\{\theta\}$ and $\Omega=\{\omega\}$, two Hilbert spaces, $\mathcal{H}_S$ and $\mathcal{H}_R$, and two ensembles, $\mathcal{E}=\{\rho_S^\theta\}_{\theta\in\Theta}$ and $\mathcal{F}=\{\tau_R^\omega\}_{\omega\in\Omega}$. Then we denote as $\mathcal{F}\otimes\mathcal{E}$ the ensemble $\{\tau_R^\omega\otimes\rho_S^\theta\}_{\omega\in\Omega,\theta\in\Theta}$
- clearly, $\mathcal{F}\otimes\mathcal{E}$ is itself an ensemble with parameter set $\Omega\times\Theta$ and Hilbert space $\mathcal{H}_R\otimes\mathcal{H}_S$
- with $\mathcal{F}\otimes\mathcal{E}$, we can play extended statistical games $(\Omega\times\Theta,\mathscr{U},\ell)$ with $\ell(\omega,\theta;u)\in\mathbb{R}$; the interpretation does not change
- we have, for example,

$$\mathbb{E}_\ell[\mathcal{F}\otimes\mathcal{E}]=\max_{\{P^u_{RS}\}}\sum_{u,\omega,\theta}\ell(\omega,\theta;u)\frac{\mathrm{Tr}\left[\left(\tau_R^\omega\otimes\rho_S^\theta\right)\ P^u_{RS}\right]}{|\Omega|\cdot|\Theta|}$$

- as a diagram:

$$\begin{array}{ccccccc} \Omega\times\Theta & \xrightarrow{\text{ensemble}} & \mathcal{H}_R\otimes\mathcal{H}_S & \xrightarrow{\text{POVM}} & \mathscr{U} & & \\ \wr & & \wr & & \wr & \Longrightarrow & \ell(\omega,\theta;u) \\ (\omega,\theta) & \xrightarrow[\mathcal{F}\otimes\mathcal{E}]{} & \tau_R^\omega\otimes\rho_S^\theta & \xrightarrow[\{P^u_{RS}\}]{} & u & & \end{array}$$

Quantum Blackwell's Theorem (Fully Quantum Version)

- ✔ **Extended comparison.** Given two ensembles $\mathcal{E} = \{\rho_S^\theta\}$ and $\mathcal{E}' = \{\sigma_{S'}^\theta\}$, we can supplement them both with the same extra ensemble $\mathcal{F} = \{\tau_R^\omega\}$ and play statistical games $(\Omega \times \Theta, \mathscr{U}, \ell)$.

Definition (Extended information ordering)

We say that $\mathcal{E} = \{\rho_S^\theta\}$ is completely more informative than $\mathcal{E}' = \{\sigma_{S'}^\theta\}$, in formula, $\mathcal{E} \succcurlyeq \mathcal{E}'$, if and only if $\mathbb{E}_\ell[\mathcal{F} \otimes \mathcal{E}] \geqslant \mathbb{E}_\ell[\mathcal{F} \otimes \mathcal{E}']$ for all extra ensembles $\mathcal{F} = \{\tau_R^\omega\}$ and all statistical games $(\Omega \times \Theta, \mathscr{U}, \ell)$.

- ✔ **Remark.** In the classical case, $\succcurlyeq \iff \succ$. In the quantum case, in general, only $\succcurlyeq \implies \succ$ holds (analogously to "positivity" versus "complete positivity")

Theorem (Fully Quantum Blackwell's Theorem)

$\mathcal{E} \succcurlyeq \mathcal{E}'$ if and only if there exists a quantum channel (CPTP map) $\Phi : \mathsf{L}(\mathcal{H}_S) \to \mathsf{L}(\mathcal{H}_{S'})$ such that $\sigma_{S'}^\theta = \Phi(\rho_S^\theta)$ for all $\theta \in \Theta$.

Intermediate Summary

- ✔ **Original.** Experiment $w(x|\theta)$ is more informative than experiment $w'(y|\theta)$, i.e., $w(x|\theta) \succ w'(y|\theta)$, if and only if there exists a noisy channel (conditional probability) $\varphi(y|x)$ such that $w'(y|\theta) = \sum_x \varphi(y|x) w(x|\theta)$
- ✔ **Quantum vanilla.** Ensemble $\mathcal{E} = \{\rho_S^\theta\}$ is more informative than ensemble $\mathcal{E}' = \{\sigma_{S'}^\theta\}$, i.e., $\mathcal{E} \succ \mathcal{E}'$, if and only if there exists a quantum statistical morphism $\mathcal{L}$ on $\mathcal{E}_\mathbb{C}$ such that $\mathcal{L}(\rho_S^\theta) = \sigma_{S'}^\theta$ for all $\theta \in \Theta$
- ✔ **Semiquantum.** Ensemble $\mathcal{E} = \{\rho_S^\theta\}$ is more informative than commuting ensemble $\mathcal{E}' = \{\sigma_{S'}^\theta\}$, i.e., $\mathcal{E} \succ \mathcal{E}'$, if and only if there exists a quantum channel (CPTP map) Φ such that $\Phi(\rho_S^\theta) = \sigma_{S'}^\theta$ for all $\theta \in \Theta$
- ✔ **Fully quantum.** Ensemble $\mathcal{E} = \{\rho_S^\theta\}$ is completely more informative than ensemble $\mathcal{E}' = \{\sigma_{S'}^\theta\}$, i.e., $\mathcal{E} \succcurlyeq \mathcal{E}'$, if and only if there exists a quantum channel (CPTP map) Φ such that $\Phi(\rho_S^\theta) = \sigma_{S'}^\theta$ for all $\theta \in \Theta$

Part II

Applications to Quantum Information Science

Section 1

Quantum Thermodynamics

The Binary Case, i.e. $\Theta = \{1, 2\}$

- assume that the unknown parameter has only two possible values $\Theta = \{\theta_1, \theta_2\} \equiv \{1, 2\}$
- in this case, classical statistical experiments become pairs of distributions $\{w_1(x), w_2(x)\}$, called "dichotomies"
- **Binary statistical game.** A statistical game $(\Theta, \mathscr{U}, \ell)$ with $\Theta = \mathscr{U} = \{1, 2\}$

Theorem (Blackwell's Theorem for Dichotomies)

Given two dichotomies $w = \{w_1(x), w_2(x)\}$ *and* $w' = \{w'_1(y), w'_2(y)\}$, $w \succ w'$ *if and only if* $\mathbb{E}_\ell[w] \geqslant \mathbb{E}_\ell[w']$ *for all binary statistical games.*

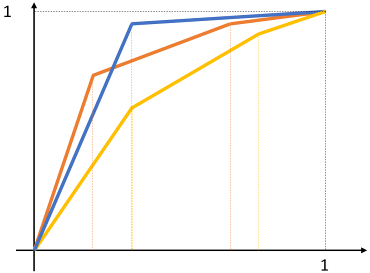
- in other words, when $\Theta = \{1, 2\}$, the "value" of a classical statistical experiment can be estimated with binary decisions:

$$\begin{array}{ccccc} \{1,2\} & \xrightarrow{\text{experiment}} & \mathscr{X} & \xrightarrow{\text{decision}} & \{1,2\} \\ \wr & & \wr & & \wr \\ \theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u \end{array} \quad \Longrightarrow \quad \ell(\theta, u)$$

- in formula: for classical dichotomies, $w \succ w' \iff w \succ_2 w'$. (The symbol $\succ_2$ denotes the information ordering restricted to binary statistical games.)

Graphical Interpretation

- given a distribution $p(x)$ denote by $p_i^\downarrow$ its i-th largest entry
- given two distributions $p(x)$ and $q(x)$, define $L(p,q)$ to be the piecewise linear curve joining the points $(x_k, y_k) = \left(\sum_{i=1}^k q_i^\downarrow, \sum_{i=1}^k p_i^\downarrow\right)$ with the origin $(0,0)$



- **Fact.** $\{p(x), q(x)\} \succ_2 \{p'(y), q'(y)\}$ if and only if $L(p,q) \geqslant L(p',q')$

Blackwell's theorem for dichotomies (reformulation)

$L(p,q) \geqslant L(p',q')$ if and only if there exists a conditional probability $\varphi(y|x)$ such that $p'(y) = \sum_x \varphi(y|x)p(x)$ and $q'(y) = \sum_x \varphi(y|x)q(x)$

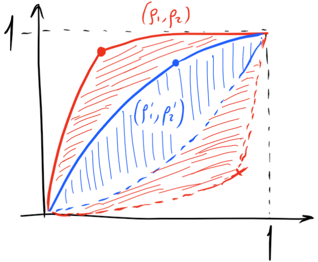
- if $q = q' = e$ uniform distribution: Lorenz curves and majorization
- if $q = q' = g$ Gibbs (thermal) distribution: thermomajorization

Quantum Lorenz Curve

- ✔ we saw that the ordering $\succ_2$ is described by Lorenz curves
- ✔ how does the ordering $\succ_2$ look like for quantum dichotomies?

Definition (Quantum Lorenz Curve)

Given a binary ensemble $\mathcal{E} = \{\rho_1, \rho_2\}$, define the curve $L(\rho_1, \rho_2)$ as the upper boundary of the region $\mathcal{R}(\rho_1, \rho_2) \triangleq \{(x, y) = (\mathrm{Tr}[E\ \rho_2], \mathrm{Tr}[E\ \rho_1]) : 0 \leqslant E \leqslant \mathbb{1}\}$

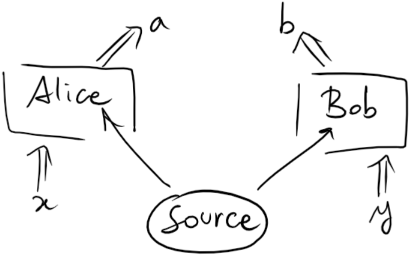


- ✔ **Fact 1.** Given two quantum dichotomies $\mathcal{E} = \{\rho_1, \rho_2\}$ and $\mathcal{E}' = \{\rho_1', \rho_2'\}$, $\mathcal{E} \succ_2 \mathcal{E}'$ if and only if $L(\rho_1, \rho_2) \geqslant L(\rho_1', \rho_2')$
- ✔ **Fact 2.** A result by Alberti and Uhlmann (1980) implies that, if both quantum ensembles are on $\mathbb{C}^2$, then $L(\rho_1, \rho_2) \geqslant L(\rho_1', \rho_2')$ if and only if there exists a CPTP map Φ such that $\Phi(\rho_i) = \rho_i'$ for $i = 1, 2$

Section 2

Entanglement and Quantum Nonlocality

Nonlocal Games



- a nonlocal game (Bell inequality) is a bipartite decision problem played "in parallel" by space-like separated players; it is formally given as $G = (\mathscr{X}, \mathscr{Y}; \mathscr{A}, \mathscr{B}; \ell)$
- **Classical source.** $p_c(a,b|x,y) = \sum_\lambda d_A(a|x,\lambda) d_B(b|y,\lambda)\pi(\lambda)$
- **Quantum source.** $p_q(a,b|x,y) = \mathrm{Tr}\left[\rho_{AB}\ (P_A^{a|x} \otimes Q_B^{b|y})\right]$
- **Expected payoff.**

$$\mathbb{E}_G[\rho_{AB}] \triangleq \max_{\{P_A^{a|x}\},\{Q_B^{b|y}\}} \sum_{x,y,a,b} \ell(x,y;a,b) p_q(a,b|x,y) \frac{1}{|\mathscr{X}|}\frac{1}{|\mathscr{Y}|}$$

- **Classical value.**

$$\mathbb{E}_G^{cl} \triangleq \max_{d_A(a|x), d_B(b|y)} \sum_{x,y,a,b} \ell(x,y;a,b) d_A(a|x) d_B(b|y) \frac{1}{|\mathscr{X}|}\frac{1}{|\mathscr{Y}|}$$

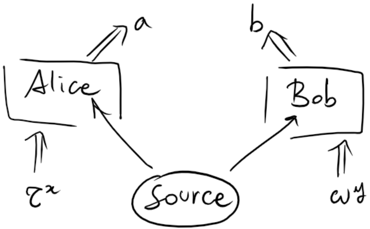
Comparison of Bipartite Quantum States

- ✔ consider two bipartite density operators: ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\sigma_{A'B'}$ on $\mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$

Definition (Nonlocality Ordering)

We say that ρ_{AB} is more nonlocal than $\sigma_{A'B'}$, in formula, $\rho_{AB} \succ_{nl} \sigma_{A'B'}$, if and only if $\mathbb{E}_G[\rho_{AB}] \geqslant \mathbb{E}_G[\sigma_{A'B'}]$ for all nonlocal games $G = (\mathscr{X}, \mathscr{Y}; \mathscr{A}, \mathscr{B}; \ell)$.

- ✔ in other words, ρ_{AB} allows to violate any Bell inequality at least as much as $\sigma_{A'B'}$ does
- ✔ can we prove a Blackwell theorem for bipartite quantum states? what does the condition $\rho_{AB} \succ_{nl} \sigma_{A'B'}$ imply about the existence of a transformation from ρ_{AB} into $\sigma_{A'B'}$?
- ✔ unfortunately not much, because of a phenomenon called...
- ✔ **Hidden Nonlocality.** Werner (1989) showed that there exist entangled bipartite states that do not exceed the classical value, for all possible Bell inequalities



- a quantum nonlocal game is given by $\Gamma = \{\mathscr{X}, \mathscr{Y}, \{\tau^x_{\tilde{A}}\}, \{\omega^y_{\tilde{B}}\}; \mathscr{A}, \mathscr{B}; \ell\}$
- **Expected payoff.**

$$\mathbb{E}_\Gamma[\rho_{AB}] \triangleq \max_{\{P^a_{\tilde{A}A}\},\{Q^b_{B\tilde{B}}\}} \sum_{x,y,a,b} \ell(x,y;a,b) \frac{\mathrm{Tr}\left[\left(\tau^x_{\tilde{A}} \otimes \rho_{AB} \otimes \omega^y_{\tilde{B}}\right)\left(P^a_{\tilde{A}A} \otimes Q^b_{B\tilde{B}}\right)\right]}{|\mathscr{X}| \cdot |\mathscr{Y}|}$$

Theorem (Blackwell's Theorem for Bipartite Quantum States)

$\mathbb{E}_\Gamma[\rho_{AB}] \geqslant \mathbb{E}_\Gamma[\sigma_{A'B'}]$ *for all quantum nonlocal games* Γ *if and only if there exist CPTP maps* $\Phi^i_{A\to A'}$ *and* $\Psi^i_{B\to B'}$ *such that* $\sigma_{A'B'} = \sum_i p(i)(\Phi^i_A \otimes \Psi^i_B)(\rho_{AB})$

- **Remark.** Such transformations are called "local operations with shared randomness" (LORS)
- application: measurement-device independent entanglement witnesses (MDIEW)

Section 3

Open Systems Dynamics

Background: Communication Games

- recall: a statistical game is as follows

$$\begin{array}{ccccccc} \Theta & \xrightarrow{\text{experiment}} & \mathscr{X} & \xrightarrow{\text{decision}} & \mathscr{U} & & \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & \Longrightarrow & \ell(\theta,u) \\ \theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u & & \end{array}$$

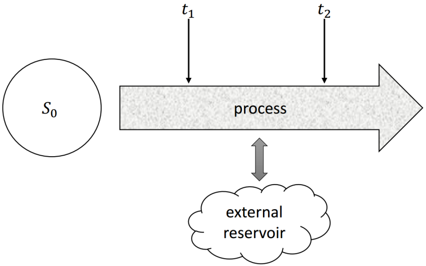
- we also interpreted θ as the input to the channel, x as the output, and u as the decoded message
- let's add the encoding into the picture:

$$\begin{array}{ccccccc} \mathscr{U} & \xrightarrow{\text{encoding}} & \Theta & \xrightarrow{\text{channel}} & \mathscr{X} & \xrightarrow{\text{decoding}} & \mathscr{U} \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ u & \xrightarrow[e(\theta|u)]{} & \theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(\hat{u}|x)]{} & \hat{u} \end{array}$$

- a communication game is a triple $(\mathscr{U},\Theta,e(\theta|u))$ and the payoff is the probability of guessing the message correctly:

$$P^e_{\text{guess}}[w] \triangleq \max_{d(\hat{u}|x)} \sum_{u,\theta,x} d(u|x)w(x|\theta)e(\theta|u)\frac{1}{|\mathscr{U}|}$$

Divisible Evolutions



- ✔ a system S is prepared at time t_0 and put in contact with an external reservoir (i.e., the environment); consider two snapshots at times $t_1 \geqslant t_0$ and $t_2 \geqslant t_1$
- ✔ two channels, w_1 and w_2, describe the evolution of the system from t_0 to t_1 and from t_0 to t_2, respectively
- ✔ the evolution from $t_0 \to t_1 \to t_2$ is physically divisible (or "memoryless") whenever there exists another channel φ such that $w_2 = \varphi \circ w_1$

Theorem (Blackwell's Theorem for Open Systems Dynamics)

The evolution $t_0 \to t_1 \to t_2$ is divisible if and only if $P^e_{\text{guess}}[w_1] \geqslant P^e_{\text{guess}}[w_2]$ for all communication games $(\mathscr{U}, \Theta, e(\theta|u))$

- ✔ for the quantum case, see the references for further details

Essential Bibliography

(this list is not meant to be an exhaustive bibliography, but only a selection of accessible, introductory, mostly self-contained texts on the topics covered in this lecture)

General theory:

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